Abstract

In this paper we consider discrete-time nonlinear systems that are affected, possibly simultaneously, by parametric uncertainties and disturbance inputs. The min-max model predictive control (MPC) methodology is employed to obtain a controller that robustly steers the state of the system towards a desired equilibrium. The aim is to provide a priori sufficient conditions for robust stability of the resulting closed-loop system via the input-to-state stability framework. First, we show that only input-to-state practical stability can be ensured in general for perturbed nonlinear systems in closed-loop with min-max MPC schemes and we provide explicit bounds on the evolution of the closed-loop system state. Then, we derive new sufficient conditions that guarantee input-to-state stability of the min-max MPC closed-loop system, via a dual-mode approach.
On the Stability of Min-max Nonlinear Model Predictive Control

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Abstract—In this paper we consider discrete-time nonlinear systems that are affected, possibly simultaneously, by parametric uncertainties and additive disturbance inputs. The min-max Model Predictive Control (MPC) methodology is employed to obtain a controller that robustly steers the state of the system towards a desired equilibrium. The aim is to provide a priori sufficient conditions for stability of the resulting closed-loop system. The novelty of the proposed approach consists in using the Input-to-State Stability (ISS) framework. First, we show that only input-to-state practical stability can be guaranteed in general for perturbed nonlinear systems in closed-loop with min-max MPC schemes. Then, we prove that under an additional assumption ISS can be achieved for the closed-loop system, via a dual-mode approach.

Keywords—Min-max, Nonlinear model predictive control, Input-to-state stability, Input-to-state practical stability.

I. INTRODUCTION

The problem of robustly regulating discrete-time nonlinear systems affected, possibly simultaneously, by parametric uncertainties and additive disturbance inputs towards a desired equilibrium has attracted the interest of many researchers, since this situation is often encountered in practical control applications. In the case when hard constraints are imposed on state and input variables, the robust Model Predictive Control (MPC) methodology provides a reliable solution for addressing this control problem. In robust MPC, the control input is obtained by solving at each sampling instant a min-max problem (originally proposed by Witsenhausen [1]), which optimizes robust performance (the minimum over the control input and the maximum over the disturbance input) while enforcing input and state constraints for all possible disturbance realizations. The research related to robust MPC is focused on solving efficiently min-max MPC optimization problems on one hand, and guaranteeing stability of the controlled system, on the other hand. In this paper we are interested in the stability problem. See [2] and the references therein for a complete view, also regarding computational aspects.

There are several ways for designing robust MPC controllers for perturbed nonlinear systems. One way is to rely on the inherent robustness properties of nominally stabilizing nonlinear MPC algorithms, e.g. as it was done [3]–[6]. Another approach is to incorporate knowledge about the disturbance in the MPC problem formulation via open-loop worst case scenarios, e.g. as done in [7]. However, the open-loop approach is likely to result in a very small set of feasible states, which makes it conservative. As a remedy, the closed-loop or feedback min-max MPC problem set-up was introduced in [8] and further developed in [9]–[11]. Sufficient conditions for asymptotic stability of nonlinear system in closed-loop with feedback min-max MPC controllers were presented in [9] for the particular case when the additive disturbance input converges to zero as time tends to infinity. These results were extended in [11] to the case when persistent additive disturbance inputs affect the system using a so-called robust stability approach.

In this paper we present a new approach to the stability problem of min-max MPC of perturbed nonlinear systems, which is based on the Input-to-State Stability (ISS) framework [12]–[14]. The first contribution of this paper is to show that only Input-to-State Practical Stability (ISpS) [15]–[17] can be ensured in general for min-max nonlinear MPC, due to the fact that the min-max MPC value function, which is usually used as the candidate Lyapunov (ISS) function, never converges to zero. This is because the min-max MPC controller takes into account the effect of a non-zero disturbance input, even if the additive disturbance input vanishes in reality. In other words, ISpS does not imply asymptotic stability for zero disturbance inputs, as it is the case for ISS. However, we prove that the developed ISpS sufficient conditions actually imply that the state trajectory of the closed-loop system is ultimately bounded in a Robustly Positively Invariant (RPI) set [18], which is the second contribution of this paper.

Still, in the case when the additive disturbance input converges to zero, it is desirable that asymptotic stability is recovered for the controlled system, i.e. the state also converges to zero. Therefore, we employ a dual-mode approach in order to obtain sufficient conditions for ISS of feedback min-max nonlinear MPC, which is the main result of this paper. ISS is proven for the dual-mode min-max MPC algorithm using a new technique based on $\mathcal{KL}$ estimates of stability, e.g. see [19].

The paper is organized as follows. Section II presents the notation used, while the ISpS (ISS) framework is introduced in Section III. The min-max MPC problem setup is briefly described in Section IV and the ISpS and ISS results for min-max nonlinear MPC are presented in Section V. Conclusions are summarized in Section VI.
II. NOTATION AND BASIC NOTIONS

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1,c_2 \in \mathbb{Z}_+$. For any real $\lambda \geq 0$, the set $\mathcal{A}_\lambda$ is defined as $\{x \in \mathbb{R}^n \mid x = \lambda y \text{ for some } y \in \mathcal{A}\}$. Let $\|\cdot\|$ denote an arbitrary p-norm. For a sequence $\{z_j\}_{j \in \mathbb{Z}_+}$ with $z_j \in \mathbb{R}$, let $\|\{z_j\}_{j \in \mathbb{Z}_+}\| := \sup\{|z_j| \mid j \in \mathbb{Z}_+\}$. Let $z_{[k]}$ denote the truncation of $\{z_j\}_{j \in \mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k]} = z_j$ if $j \leq k$, and $z_{[k]} = 0$ if $j > k$. Also, let $z_{[k_1,k_2]}$ denote the truncation of $\{z_j\}_{j \in \mathbb{Z}_+}$ at times $k_1 \in \mathbb{Z}_{\geq 1}$ and $k_2 \in \mathbb{Z}_{\geq 1}$, i.e. $z_{[k_1,k_2]} = z_j$ if $k_1 \leq j \leq k_2$, and $z_{[k_1,k_2]} = 0$ if $j < k_1$ or $j > k_2$. A convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a C-set.

III. INPUT-TO-STATE STABILITY

In this section we present the ISS framework for studying the stability properties of discrete-time perturbed autonomous nonlinear systems. These ideas will be employed in this paper to study the behavior of perturbed nonlinear systems in closed-loop with min-max MPC controllers.

Consider the discrete-time autonomous perturbed nonlinear system described by

$$x_{k+1} = G(x_k, w_k, v_k), \quad k \in \mathbb{Z}_+,$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{W} \subset \mathbb{R}^d_w$ and $v_k \in \mathcal{V} \subset \mathbb{R}^d_v$ are the state, unknown parametric uncertainties and additive disturbance inputs at discrete-time $k$ and, $G : \mathbb{R}^2 \times \mathbb{R}^d_v \times \mathbb{R}^d_w \rightarrow \mathbb{R}^n$ is an arbitrary nonlinear function. We assume that $\mathbb{W}$ is a known compact set and $\mathcal{V}$ is a known C-set.

For simplicity of notation, we assume that the origin is an equilibrium in (1) for zero additive disturbance, meaning that $G(0, w, 0) = 0$ for all $w \in \mathbb{W}$.

Definition III.1 For a given $0 \leq \lambda \leq 1$, a set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a (robustly) $\lambda$-contractive set for system (1) if for all $x \in \mathcal{P}$ it holds that $G(x, w, v) \in \lambda \mathcal{P}$ for all $w \in \mathbb{W}$ and all $v \in \mathcal{V}$.

Definition III.2 Let $\mathcal{X} \subseteq \mathbb{R}^n$. The system (1) is said to be Ultimately Bounded (UB) in a set $\mathcal{P} \subset \mathbb{R}^n$ for initial conditions in $\mathcal{X}$, if for all $x_0 \in \mathcal{X}$ there exists a $p \in \mathbb{Z}_+$ such that for all $k \in \mathbb{Z}_{\geq P}$, $w_{[k-1]} \in \mathbb{W}^k$ and all $v_{[k-1]} \in \mathbb{V}^k$ it holds that $x_k \in \mathcal{P}$.

Definition III.3 A real-valued scalar function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\phi(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if $\phi \in \mathcal{K}$ and it is radially unbounded (i.e. $\phi(s) \rightarrow \infty$ as $s \rightarrow \infty$). A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if for each fixed $k$, $\beta(\cdot,k) \in \mathcal{K}_\infty$ and for each fixed $s$, $\beta(s, \cdot) \in \mathcal{K}$ is non-increasing and $\lim_{t \rightarrow \infty} \beta(s, t) = 0$.

Next, we define the notions of Input-to-State practical Stability (ISpS) [15]–[17] and Input-to-State Stability (ISS) [12]–[14] for the discrete-time perturbed system (1).

Definition III.4 The system (1) is said to be ISpS for initial conditions in $\mathcal{X}$ if there exist a $\mathcal{KL}$-function $\beta$, a $\mathcal{K}$-function $\gamma$ and a non-negative constant $d$ such that, for each $x_0 \in \mathcal{X}$, each $w_{[k-1]} \in \mathbb{W}^k$ and each $v_{[k-1]} \in \mathbb{V}^k$ it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) + d, \quad \forall k \in \mathbb{Z}_{\geq 1}.$$

(2)

If (2) is satisfied with $d = 0$ for each $x_0 \in \mathcal{X}$, each $w_{[k-1]} \in \mathbb{W}^k$ and each $v_{[k-1]} \in \mathbb{V}^k$, the system (1) is said to be ISS for initial conditions in $\mathcal{X}$.

The ISS property guarantees both ultimate boundedness in the presence of persistent disturbances and robust asymptotic stability in the presence of decaying uncertainties. In this paper we prove that for feedback min-max MPC is only possible to guarantee ISpS for the resulting closed-loop system and, in that case, only ultimate boundedness can be assured.

In what follows we state a version of the continuous-time ISpS sufficient conditions of Proposition 2.1 of [17], applied for the constrained discrete-time perturbed system (1). This result will be used throughout the paper to prove ISpS and ISS for the particular case of min-max nonlinear MPC.

Theorem III.5 Let $d_1, d_2$ be non-negative constants, let $a,b,c,\lambda$ be positive constants with $c \leq b$ and let $\alpha_i(s) := as^i$, $\beta_i(s) := bs^i$, $\alpha_3(s) := cs^i$ and $\sigma \in \mathcal{K}$. Furthermore, let $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a function and let $\mathcal{X}$ be a RPI set for system (1) and disturbances $w$ in $\mathbb{W}$ and $v$ in $\mathbb{V}$. Suppose that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + d_1$$

(3a)

$$V(G(x,w,v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2$$

(3b)

for all $x \in \mathcal{X}$, all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. Then it holds that:

(i) The system (1) is ISpS for initial conditions in $\mathcal{X}$;

(ii) If the inequalities (3) hold with $d_1 = d_2 = 0$, the system (1) is ISS for initial conditions in $\mathcal{X}$.

Proof: (i) From the hypothesis we have that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + d_1$$

for all $x \in \mathcal{X}$, where $\rho := 1 - \frac{c}{b} \in [0,1]$. Since $V(0) - \alpha_2(0) = V(0) = \rho V(0) + (1 - \rho) d_1 \leq \rho V(0) + (1 - \rho) d_1$ for any $\rho \in [0,1]$, we have that $V(x) - \alpha_2(\|x\|) \leq \rho V(x) + (1 - \rho) d_1$, for all $x \in \mathcal{X}$ \{0\}, where $\rho := 1 - \frac{c}{b} \in [0,1]$. Since $V(0) - \alpha_2(0) = V(0) = \rho V(0) + (1 - \rho) d_1 \leq \rho V(0) + (1 - \rho) d_1$ for any $\rho \in [0,1]$, we have that $V(x) - \alpha_2(\|x\|) \leq \rho V(x) + (1 - \rho) d_1$, for all $x \in \mathcal{X}$. Then, inequality (3b) becomes

$$V(G(x,w,v)) \leq \rho V(x) + \sigma(\|v\|) + (1 - \rho) d_1 + d_2.$$
for all $x \in X$, $w \in W$ and all $v \in V$. Due to robust positive invariance of $X$, inequality (4) yields repetitively:

$$V(x_{k+1}) \leq \rho^{k+1}V(x_0) + \rho^k(\sigma(||v||) + (1 - \rho)d_1 + d_2) + \rho^{k-1}(\sigma(||v||) + (1 - \rho)d_1 + d_2) + \ldots + \sigma(||v||) + (1 - \rho)d_1 + d_2,$$

for all $x_0 \in X$, $w \in W$, $v \in V$, $k \in \mathbb{Z}_+$. Then, it follows that:

$$\alpha_1(||x_0||) \leq V(x_0) \leq \rho^{k+1}\alpha_0(||x_0||) + \rho^{k+1}d_1 + \sum_{i=0}^{k} \rho^i(\sigma(||v_i||) + (1 - \rho)d_1 + d_2) \leq \rho^{k+1}\alpha_2(||x_0||) + \rho^i(\sigma(||v_i||) + (1 - \rho)d_1 + d_2) \leq \rho^{k-1}\alpha_2(||x_0||) + (1 + \rho)d_1 + \frac{d_2}{1 - \rho} + \sigma(||v||),$$

for all $x_0 \in X$, $w[k] \in W^{k+1}$, $v[k] \in V^{k+1}$, $k \in \mathbb{Z}_+$. Let $\theta := (1 + \rho)d_1 + \frac{d_2}{1 - \rho}$. As $\alpha_1 \in \mathcal{A}_\rho$ and $\sigma \in \mathcal{A}$, it follows that:

$$\|x_{k+1}\| \leq \alpha_1^{-1}\left(\rho^{k+1}\alpha_2(||x_0||) + \theta + \sigma(||v||)\right) \leq \alpha_1^{-1}\left(2\max\left(\rho^{k+1}\alpha_2(||x_0||) + \theta, \sigma(||v||)\right)\right) \leq \alpha_1^{-1}(2\rho^{k+1}\alpha_2(||x_0||) + \theta) + \alpha_1^{-1}(2\sigma(||v||)) \leq \alpha_1^{-1}(4\rho^{k+1}\alpha_2(||x_0||)) + \alpha_1^{-1}(2\sigma(||v||)) + \alpha_1^{-1}(4\theta),$$

for all $x_0 \in X$, $w[k] \in W^{k+1}$, $v[k] \in V^{k+1}$, $k \in \mathbb{Z}_+$. Let $\beta(s,k) := \alpha_1^{-1}(4\rho^k\alpha_2(s))$. For a fixed $k \in \mathbb{Z}_+$, we have that $\beta(s,k) \in \mathcal{A}$ due to $\alpha_2 \in \mathcal{A}_\rho$, $\alpha_1^{-1} \in \mathcal{A}_\rho$ and $\rho \in (0,1)$. For a fixed $s$, it follows that $\beta(s,\cdot)$ is non-increasing and $\lim_{k \to \infty} \beta(s,k) = 0$, due to $\rho \in (0,1)$ and $\alpha_1^{-1} \in \mathcal{A}_\rho$. Thus, it follows that $\beta \in \mathcal{A}_\rho$. Now let $\gamma(s) := \alpha_1^{-1}(2\rho^k\alpha_2(s))$. Since $\frac{1}{1 - \rho} > 0$, it follows that $\gamma \in \mathcal{A}_\rho$ due to $\alpha_1^{-1} \in \mathcal{A}_\rho$ and $\sigma \in \mathcal{A}_\rho$. Finally, let $d := \alpha_1^{-1}(4\theta)$. Since $\rho \in (0,1)$ and $d_1, d_2 \geq 0$, we have that $d \geq 0$. Hence, the perturbed system (1) is ISS in the sense of Definition III.4 for initial conditions $x$. (ii) The proof is analogue with the proof of part (i) with the observation that the ISS property of Definition III.4 holds with $\beta(s,k) := \alpha_1^{-1}(2\rho^k\alpha_2(s))$ and we do not need to assume that $c \leq b$ in this case. For a complete proof of part (ii), the reader is referred to [20].

We call a function $V$ that satisfies the hypothesis of Theorem III.5 (with $d_1 = d_2 = 0$) and ISS (ISS) function. Note that the hypothesis of Theorem III.5 does not require continuity of $V$ nor that $V(0) = 0$. This makes the ISS framework suitable for analyzing stability of nonlinear systems in closed-loop with min-max MPC controllers, as explained in the introduction.

\[\text{IV. FEEDBACK MIN-MAX NONLINEAR MPC: PROBLEM SET-UP}\]

Consider the discrete-time non-autonomous perturbed nonlinear system described by

$$x_{k+1} = g(x_k, u_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \quad (5)$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $w_k \in \mathbb{W} \subset \mathbb{R}^{d_W}$ and $v_k \in \mathbb{V} \subset \mathbb{R}^{d_V}$ are the state, unknown parametric uncertainties and additive disturbance inputs at discrete-time $k$ and $g: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{d_W} \times \mathbb{R}^{d_V} \to \mathbb{R}^n$ is an arbitrary nonlinear function with $g(0,0,0,0) = 0$ for all $w \in \mathbb{W}$. Let $X \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ be $\mathcal{C}$-sets that implement state and input constraints for system (5). Let $F: \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ with $L(0,0) = 0$ be convex functions.

Feedback min-max MPC obtains a sequence of feedback control laws that minimizes a worst case cost function, while assuring robust constraint handling. In this paper we employ the dynamic programming approach to feedback min-max nonlinear MPC proposed in [8] for linear systems and in [9] for nonlinear systems. Let $\mathbb{X}_T \subset \mathbb{X}$ denote a desired target set that contains the origin. The set of feasible states that can be robustly controlled into the target set $\mathbb{X}_T$ in $i \in \mathbb{Z}_+ \leq \mathbb{X}$ steps or less is defined as:

$$\mathbb{X}_f(i) := \{x \in \mathbb{X} | \exists u \in \mathbb{U} \text{ s.t. } x_k \in \mathbb{X}_f(i), x_k \in \mathbb{X}_T \},$$

for all $k \in \mathbb{Z}_+$ and $x \in \mathbb{X}_f(i)$. Then, we prove that ISS can be ensured in $\mathbb{X}_f(i) \to \mathbb{R}_+$, $i \in \mathbb{Z}_+$...
h(x) ∈ U} denote the safe set with respect to both state and input constraints for h.

**Assumption V.1** There exist a, b, λ > 0 with a ≤ b, non-negative constants e₁, e₂ and a \( \mathcal{H} \)-function \( \sigma \) such that:
1. \( X_\tau \subseteq X_U \);
2. \( X_\tau \) is a RPI set for system (5) in closed-loop with \( h(x) \);
3. \( L(x,u) \geq a \|x\|^3 \) for all \( x \in X \) and all \( u \in U \);
4. \( F(x) \leq b \|x\|^2 + e_1 \) for all \( x \in X_\tau \);
5. \( F(g(x,h(x),w,v)) - F(x) \leq -L(x,h(x)) + \sigma(\|v\|) + e_2 \) for all \( x \in X_\tau \), all \( w \in W \) and all \( v \in V \).

The next result is directly obtained via Theorem III.5 by showing that the terminal cost \( F \) is a local cost (i.e. for all \( x \in X_\tau \)) ISpS (ISS) function for system (5) in closed-loop with \( h(x) \).

**Proposition V.2** Suppose that Assumption V.1 holds (with \( e_1 = e_2 = 0 \)) and there exist positive constants \( a_1, \lambda \) such that \( F(x) \geq a_1 \|x\|^3 \) for all \( x \in X_\tau \). Then, system (5) in closed-loop with \( h(x) \) is ISpS (ISS) for initial conditions in \( X_\tau \).

**Theorem V.3** Suppose that \( F, L, X_\tau \) and \( h \) are such that Assumption V.1 holds for system (5). Then, the perturbed nonlinear system (5) in closed-loop with the feedback min-max MPC control (6) is ISpS for initial conditions in \( X_f(N) \).

Proof: The proof consists in showing that the min-max MPC value function \( V \) is an ISpS function, i.e. it satisfies the hypotheses of Theorem III.5. Note that it is known [9, 21] that the set \( X_f(N) \) is a RPI set for system (5) in closed-loop with \( \bar{u}(x_k), k \in Z_+ \). From Assumption V.1-3) it follows that \( V(x) = V(x_0) \geq L(x, \bar{u}(x)) \geq a \|x\|^3 \), for all \( x \in X_f(N) \). Next, let \( x_0 := x \in X_\tau \), from Assumption V.1-2,5) it follows that for any \( w_\nu \in \mathbb{W}^R \) and any \( v_\nu \in \mathbb{V}^R \):

\[
F(x_0) + \sum_{i=0}^{N-1} L(x_i,h(x_i)) \leq F(x_0) + \sum_{i=0}^{N-1} \sigma(\|v\|) + NE_{e_2},
\]

where \( x_i := g(x_{i-1}, h(x_{i-1}), w_{i-1}, v_{i-1}) \) for \( i = 1, \ldots, N \). Then, by optimality and Assumption V.1-4) we have that for all \( x \in X_\tau \):

\[
V(x) = V_0(x) \leq F(x) + N(\max_{v \in V} \sigma(\|v\|) + e_2) \leq b \|x\|^2 + d_1,
\]

where \( d_1 := e_1 + N(\max_{v \in V} \sigma(\|v\|) + e_2) > 0 \). Next, we employ the reasoning of Lemma 4 of [11] to establish a global upper bound on \( V \). Let \( r > 0 \) be such that \( \mathcal{B}_r := \{ x \in \mathbb{R}^n \mid \|x\| \leq r \} \subseteq X_\tau \). Due to compactness of \( X, U, W, V \) and convexity of \( F, L \), there exists a constant \( \Gamma > 0 \) such that \( V(x) \leq \Gamma \) for all \( x \in X_f(N) \). Letting \( \theta := \max(1, \frac{r}{\sqrt{n}}) \) we obtain \( V(x) \leq \theta \|x\|^2 \) for all \( x \in X_f(N) \) \( \setminus X_\tau \). Then, due to \( \theta \geq 1 \) it follows that \( V(x) = V_0(x) \leq \theta b \|x\|^2 + d_1 \) for all \( x \in X_f(N) \). Hence, \( V \) satisfies the condition (3a) for all \( x \in X_f(N) \).

Next, we show that \( V \) satisfies the condition (3b). By Assumption V.1-5) we have that:

\[
V_1(x) - V_0(x) = \min_{w \in W} \left\{ \max_{v \in V} [L(x,u) + F(g(x,u,w,v))] \right\}
\]

subject to \( g(x,u,w,v) \in X_\tau, \forall w \in W, \forall v \in V \) \( - F(x) \leq \max_{w \in W, v \in V} [L(x,h(x)) + F(g(x,u,w,v))] - F(x) \leq \max_{v \in V} \sigma(\|v\|) + e_2.
\]

Then, using the same reasoning as the one used in the proof of Theorem 2 of [11] we obtain via induction that:

\[
V_{i+1}(x) - V_i(x) \leq \max_{v \in V} \sigma(\|v\|) + e_2, \forall x \in X_f(i), \forall i \in Z_+.
\]

(7)

At time \( k \in Z_+ \), for a given state \( x_k \in X \) and a fixed prediction horizon \( N \) the min-max MPC control law \( \bar{u}(x_k) \) is calculated and then applied to system (5). The state evolves to \( x_{k+1} = g(x_k, \bar{u}(x_k), w_k, v_k) \in X_f(N) \). Then, by Assumption V.1-5) and applying recursively (7) it follows that:

\[
V(x_{k+1}) - V(x_k) = V_N(x_{k+1}) - V_N(x_k) \leq V_N(x_{k+1}) - V_N \left( \max_{w \in W, v \in V} [L(x_k, u_k)] \right)
\]

subject to \( g(x_k, u_k, w, v) \in X_f(N-1), \forall w \in W, \forall v \in V \) \( V_N(x_{k+1}) = \max_{w \in W, v \in V} [L(x_k, \bar{u}(x_k)) + V_{N-1}(g(x_k, \bar{u}(x_k), w_k, v_k)) \leq V_N(x_{k+1}) - L(x_k, \bar{u}(x_k)) - V_{N-1}(g(x_k, \bar{u}(x_k), w_k, v_k)) \leq -L(x_k, \bar{u}(x_k)) + \max_{v \in V} \sigma(\|v\|) + e_2 \leq -a \|x_k\|^2 + \max_{v \in V} \sigma(\|v\|) + e_2 \leq -a \|x_k\|^2 + \sigma(\|v\|) + d_2,\)

(8)

for all \( x_k \in X_f(N), w_k \in W, v_k \in V \) and all \( k \in Z_+ \), where \( d_2 := \max_{v \in V} \sigma(\|v\|) + e_2 \). Hence, the feedback min-max nonlinear MPC value function \( V \) satisfies the ISpS sufficient conditions of Theorem III.5. The statement then follows from Theorem III.5.

**A. ISS Dual-mode Min-max Nonlinear MPC**

Although the perturbed system (5) may be input-to-state stabilizable, ISS cannot be proven for system (5) in closed-loop with \( \bar{u} \) if the MPC value function \( V \) is used as the candidate ISS function. As shown in the proof of Theorem V.3, this is due to the fact that by construction, \( V \) satisfies the conditions (3) with \( d_1, d_2 > 0 \), even if Assumption V.1 holds with \( e_1 = e_2 = 0 \). In the case of persistent disturbances this is not necessarily a drawback, since ultimate boundedness is achieved due to robust positive invariance of \( X_f(N) \). However, in the case when the disturbance vanishes after a certain time it is desirable to have an ISS closed-loop system, since then ISS implies asymptotic stability.

Next, we present sufficient conditions for ISS of system (5) in closed-loop with a dual-mode min-max MPC
strategy. These conditions do not involve an MPC value function equal to zero in the terminal set $\mathcal{X}_T$.

The following technical result will be employed to prove the main result for dual-mode min-max nonlinear MPC. Let

$$\mathcal{B}_1(\mathcal{P}) := \{g(x, \bar{u}(x), w, v) \mid x \in \mathcal{P}, w \in \mathbb{W}, v \in \mathcal{V}\}$$

be the robust one-step reachable set [18] for the closed-loop system (5)-(6) with respect to an arbitrary set $\mathcal{P}$.

**Assumption V.4**

1. There exists $\tau \in (0, a)$ such that the set

$$\mathcal{X}^\tau := \{x \in \mathbb{R}^n \mid \|x\|^2 \leq \frac{d_2}{a - \tau}\}$$

satisfies $\mathcal{F} := \mathcal{X}_f(N) \setminus \mathcal{X}^\tau \neq \emptyset$.

2. There exists $r > 0$ such that $\eta \triangleq a - \frac{d_2}{\tau^2} > 0$ and the set $\mathcal{B}_r$ satisfies $\mathcal{F} := \mathcal{X}_f(N) \setminus \mathcal{B}_r \neq \emptyset$.

**Theorem V.5** Suppose that $F$, $L$, $\mathcal{X}_T$ and $h$ are such that Assumption V.1 holds for system (5) and Assumption V.4-1) (or Assumption V.4-2)) holds. For any $x_0 \in \mathcal{F}$ let $i \in \tilde{Z}_{g, i}$ be such that $x_k \in \mathcal{F}$ for all $k \in \tilde{Z}_{g, i}$. Then it holds that

(i) There exists a $\mathcal{F}$-function $\beta$ such that the trajectory of the closed-loop system (5)-(6) satisfies $\|x_k\| \leq \beta(\|x_0\|, k)$ for all $k \in \tilde{Z}_{g, i}$;

(ii) There exists an $i^* \in \tilde{Z}_{g, 1}$ such that $x_{i^*} \in \mathcal{F}$.

**Proof:** We only present the proof for the case when Assumption V.4-1) holds. The proof for the case when Assumption V.4-2) holds is analog.

(i) As shown in the proof of Theorem V.3, the hypothesis implies that

$$\|a\|^2 \leq V(x) - \theta \|b\|^2 \|x\|^2 + d_1, \quad \forall x \in \mathcal{X}_f(N).$$

Let $\bar{r} > 0$ be such that $\mathcal{B}_{\bar{r}} \subseteq \mathcal{X}$. From $x_k \notin \mathcal{F}$ for all $k \in \tilde{Z}_{g, i}$ we have that $\|x_k\| \geq \bar{r}$ for all $k \in \tilde{Z}_{g, i}$. This yields:

$$V(x_k) \leq \theta \|b\|^2 \|x\|^2 + d_1 \left(\frac{\|x_k\|}{\bar{r}}\right) \leq (\theta \|b\|^2 + d_1 / \bar{r}^2) \|x_k\|^2,$$

$$\lor x_0 \in \mathcal{X}, \forall k \in \tilde{Z}_{g, i}.$$ 

The hypothesis also implies (see (8)) that $\|x_{k+1}\| = V(x_k) + (\theta \|b\|^2 + d_1) / \bar{r}^2$ for all $x_k \in \mathcal{F}$, $k \in \tilde{Z}_{g, i}$. Then, by Assumption V.4-1) it follows that $\|x_{k+1}\| - \|x_k\| \leq -\tau \|x_k\|^2$ for all $x_k \in \mathcal{F}$ and $k \in \tilde{Z}_{g, i}$. (alternatively, by Assumption V.4-2) we have $\|x_{k+1}\| - \|x_k\| \leq -\eta \|x_k\|^2$ for all $x_k \in \mathcal{F}$ and all $k \in \tilde{Z}_{g, i}$). Then, as shown in the proof of Theorem III.5 (see also [20] for insight), the state trajectory satisfies for all $k \in \tilde{Z}_{g, i}$

$$\|x_k\| \leq \beta(\|x_0\|, k); \quad \beta(s, k) \triangleq \alpha_k(\rho \lambda_k(s)), $$

where $\alpha_k(x) := \bar{b} \times \bar{b}^x, \bar{b} := \theta b + d_1 / \bar{r}^2, \alpha_k(x) := a \lambda_k \times \lambda_k \times \lambda_k$ and $\rho := \frac{\bar{r}}{\bar{r}} \in (0, 1)$ (or $\rho := \frac{\bar{r}}{\bar{r}} \in (0, 1)$).

(ii) The proof is by contradiction. Let $\bar{r} > \bar{r} > 0$ be such that $\mathcal{X}_f(N) \subseteq \mathcal{B}_r$. Such an $\bar{r}$ exists due to the fact that the compactness of $\mathcal{X}$ implies that $\mathcal{X}_f(N)$ is bounded. Assume that there does not exist an $i^* \in \tilde{Z}_{g, 1}$ such that $x_{i^*} \in \mathcal{F}$. Then, for all $i \in \tilde{Z}_+$ we have that

$$\|x_i\| \leq \beta(\|x_0\|, i) = \left(\frac{\bar{b} \times \bar{b}}{\bar{r}}\right) \|x_0\| (\rho \lambda_i)^i \leq \left(\frac{\bar{b} \times \bar{b}}{\bar{r}}\right) \bar{r}.$$

Since $\rho \times \bar{r} \in (0, 1)$, there exists an $i^* \in \tilde{Z}_{g, 1}$ such that

$$\left(\rho \lambda_i^i\right) \bar{r} \leq \bar{r}.$$

Then, it follows that $\|x_{i^*}\| \leq \bar{r}$, which implies $x_{i^*} \in \mathcal{B}_r \subseteq \mathcal{F}$. Hence, we reach a contradiction.

**Lemma V.6** Let

$$Y \triangleq \max_{x \in \mathcal{B}_1(\mathcal{P}) \cap \mathcal{X}_f(N)} V(x),$$

where $\mathcal{P} := \mathcal{F}$ (or $\mathcal{P} := \mathcal{B}_r$), and let

$$\mathcal{Y}_f \triangleq \{x \in \mathcal{X}_f(N) \mid V(x) \leq Y\} \subseteq \mathcal{X}_f(N).$$

Suppose that the hypothesis of Theorem V.5 holds. Then, the closed-loop system (5)-(6) is ultimately bounded in the set $\mathcal{Y}_f$ for initial conditions in $\mathcal{X}_f(N)$.

**Proof:** Since $V(x) \leq Y$ for all $x \in \mathcal{P} \cap \mathcal{X}_f(N)$, it follows that $\mathcal{P} \cap \mathcal{X}_f(N) \subseteq \mathcal{Y}_f$. Suppose that $x_0 \in \mathcal{X}_f(N) \setminus \mathcal{Y}_f$. Then, by Theorem V.5-(ii) it follows that there exists an $i \in \tilde{Z}_{g, 1}$ such that $x_i \in \mathcal{P} \cap \mathcal{X}_f(N) \subseteq \mathcal{Y}_f$. Next, we prove that $\mathcal{Y}_f$ is a RPI set for the closed-loop system (5)-(6). As shown in the proof of Theorem V.5-(i), for any $x \in \mathcal{Y}_f \setminus \mathcal{X}_f(N)$ it holds that

$$V(g(x, \bar{u}(x), w, v)) \leq \bar{V}(x) - \tau \|x\|^2 \leq V(x) \leq Y,$$

for all $w \in \mathbb{W}$ and all $v \in \mathcal{V}$. We also have that for any $x \in \mathcal{P} \cap \mathcal{X}_f(N)$, $g(x, \bar{u}(x), w, v) \in \mathcal{B}_1(\mathcal{P}) \cap \mathcal{X}_f(N)$ for all $w \in \mathbb{W}$ and all $v \in \mathcal{V}$, which yields $V(g(x, \bar{u}(x), w, v)) \leq Y$. Thus, for any $x \in \mathcal{Y}_f$, it holds that $g(x, \bar{u}(x), w, v) \in \mathcal{Y}_f$ for all $w \in \mathbb{W}$ and all $v \in \mathcal{V}$, which implies that $\mathcal{Y}_f$ is a RPI set for the closed-loop system (5)-(6). Since for any $x_0 \in \mathcal{X}_f(N) \setminus \mathcal{Y}_f$ there exists an $i \in \tilde{Z}_{g, 1}$ such that $x_i \in \mathcal{Y}_f$, it follows that the closed-loop system (5)-(6) is ultimately bounded in $\mathcal{Y}_f$.

Note that in a worst case situation, i.e. when the additive disturbance is too large and Assumption V.4 does not hold for any $\tau \in (0, a)$ (or $\tau > 0$), the closed-loop system (5)-(6) is still UB in the set $\mathcal{X}_f(N)$.

Next, let the dual-mode feedback min-max MPC control law be defined as:

$$\hat{u}_{DM}(x) \triangleq \begin{cases} \hat{u}(x) & \text{if } x \in \mathcal{X}_f(N) \setminus \mathcal{X}_T, \\ h(x) & \text{if } x \in \mathcal{X}_T. \end{cases} \quad (9)$$

**Theorem V.7** Suppose that Assumption V.1 holds with $e_1 = e_2 = 0$ for system (5), and there exist $a_1, \lambda > 0$ such that $F(x) \geq a_1 \|x\|^2$ for all $x \in \mathcal{X}_f$. Furthermore, suppose there exists $\tau \in (0, a)$ such that Assumption V.4-1) holds
and $\mathcal{X} \subseteq \mathcal{X}_T$ (or, alternatively, there exists $r > 0$ such that Assumption V.4-(2) holds and $\mathcal{B}_r \subseteq \mathcal{X}_T$). Then, the perturbed nonlinear system (5) in closed-loop with the dual-mode feedback min-max MPC control $u^{\text{DM}}$ is ISS for initial conditions in $\mathcal{X}_f(N)$.

Proof: We consider two situations: in Case 1 we assume that $x_0 \in \mathcal{X}_T$ and in Case 2 we assume that $x_0 \in \mathcal{X}_f(N) \setminus \mathcal{X}_T$. Note that in Case 2, since $\mathcal{X} \subseteq \mathcal{X}_T$ (or $\mathcal{B}_r \subseteq \mathcal{X}_T$), by Theorem V.5-(ii) there exists a $k \in \mathbb{Z}_{\geq 2}$ such that $x_k \in \mathcal{X}_T$. In Case 1, $F$ satisfies the hypothesis of Proposition V.2 with $e_1 = e_2 = 0$ and hence, the closed-loop system (5)-(9) is ISS. Then, using the reasoning employed in the proof of Theorem III.5 (see also [20] for insight), it can be shown that there exist a $\mathcal{X}$-function $\beta_1(s,k) := \alpha_1^{-1}(2p_1^2\alpha_2(s))$, with $\alpha_1(s) := a_1s^\lambda$, $\alpha_2(s) := bs^\lambda$, $p_1 := \frac{a}{b}$, and a $\mathcal{X}$-function $\gamma$ such that for all $x_0 \in \mathcal{X}_T$ the state trajectory satisfies

$$\|x_k\| \leq \beta_1(\|x_0\|, k) + \gamma(\|v_{k-1}\|), \quad \forall k \in \mathbb{Z}_{\geq 2}.$$  \hfill (10)

In Case 2, from Theorem V.5-(i) we have that there exists a $\mathcal{X}$-function $\beta_1(s,k) := \alpha_1^{-1}(\rho_1^2\alpha_2(s))$, with $\alpha_1(s) := a_1s^\lambda$, $\alpha_2(s) := bs^\lambda$, $p_2 := \frac{a}{b}$ (or $p_2 := \frac{a}{b}$), and a $\mathcal{X}$-function $\gamma$ such that for all $x_0 \in \mathcal{X}_f(N) \setminus \mathcal{X}_T$ the state trajectory satisfies

$$\|x_k\| \leq \beta_2(\|x_0\|, k), \quad \forall k \in \mathbb{Z}_{\leq p} \quad \text{and} \quad x_p \in \mathcal{X}_T.$$

Then, for all $p \in \mathbb{Z}_{\geq 1}$ and all $k \in \mathbb{Z}_{\geq p+1}$ it holds that

$$\|x_k\| \leq \beta_1(\|x_0\|, p-k) + \gamma(\|v_{k-p,k-1}\|) \leq \beta_1(\beta_2(\|x_0\|,p), p-k) + \gamma(\|v_{k-p,k-1}\|) \leq \beta_1(\|x_0\|, k) + \gamma(\|v_{k-1}\|),$$

where we used the fact that

$$\beta_1(\beta_2(s,p), k-p) = \alpha_1^{-1}
\left(2\rho_1^2\alpha_2 \left(\frac{b}{a} \right)^{\frac{\lambda}{2}} \left(\rho_2^{\frac{1}{2}}\right)^p \right) \leq \left(\frac{2bb}{a^2}\right)^{\frac{\lambda}{2}} \left(\rho_2^{\frac{1}{2}}\right)^{\frac{k}{2}} =: \beta_3(s,k),$$

and $\rho_3 := \min(p_1, p_2) \in (0,1)$. Hence, $\beta_3 \in \mathcal{X}$. Then, we have that

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{k-1}\|),$$

for all $x_0 \in \mathcal{X}_f(N)$, $v_{k-1} \in \mathbb{W}_k$, $v_{k-1} \in \mathbb{V}$ for all $k \in \mathbb{Z}_{\geq 1}$, where $\beta(s,k) := \max(\beta_1(s,k), \beta_2(s,k), \beta_3(s,k))$. Since $\beta_{1,2,3}(s,k) \in \mathcal{X} \cup \mathcal{L}$, it is ISS.

VI. CONCLUSIONS

In this paper we have revisited the stability problem of min-max nonlinear Model Predictive Control. The Input-to-State Practical Stability framework has been proposed as a new approach for ensuring stability of perturbed nonlinear systems in closed-loop with min-max MPC controllers. A priori sufficient conditions for ISS were presented. Moreover, it was proven that these conditions also ensure ultimate boundedness. Then, sufficient conditions under which ISS can be achieved in min-max nonlinear MPC were derived via a dual-mode approach. This result is important because it guarantees both ultimate boundedness in the presence of persistent disturbances and robust asymptotic stability in the presence of decaying uncertainties.

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