A FAMILY OF PUMPING-DAMPING SMOOTH STRATEGIES FOR SWINGING UP A PENDULUM

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Abstract: In this paper we present some additional results regarding the pumping-damping strategy for swinging up a pendulum introduced in (Åström *et al.*, 2005). Here, the family of energy functions is enlarged and the corresponding pumping-damping functions are proposed giving rise to new smooth controllers that swing up and stabilize the pendulum. Furthermore, a generalization of the stability criterion is introduced for this larger class of controllers. *Copyright © 2006 IFAC.*

Keywords: Pendulum, Shaping Hamiltonians, Swing-up, Energy management.

1. INTRODUCTION

The family of the inverted pendula has attracted the attention of control researchers in recent decades as a benchmark for testing and evaluating a wide range of classical and contemporary nonlinear control methods. As it is well known the inverted pendulum displays two main problems: swinging up the pendulum to the upright position and stabilizing it in this position once it is reached. The classical solution to this problem is given in (Åström and Furuta, 2000), and is based on energy considerations. However, this solution is a hybrid one. A main challenge for the control community has been to find a single smooth law that copes with the global problem of leading the pendulum from the hanging position (or any other initial state, both in position and velocity) to the upright one. A solution to that problem is given in (Åström *et al.*, 2005). In the present paper we generalize the results to a larger class of energy functions and pumping-damping strategies.

We consider the simplest version of the pendulum (Åström and Furuta, 2000): the control action is the acceleration of the pivot and, thus, a 2-dimensional model is used. Here, we return to the idea of (Åström *et al.*, 2005). First, an energy shaping control law is designed in such a way that: 1) the closed-loop energy presents a minimum at the desired position; and 2) the energy shaping controller is globally defined. Since the chosen target energy has other minima different than the desired equilibrium, a combination of energy dissipation (damping) and injection (pumping) is needed in order to globally stabilize the origin. To that end an oval closed curve circumscribing about the region where pumping is needed was introduced in (Åström *et al.*, 2005) for a quite simple case. The resultant law is smooth, no commutations are needed, and the origin of the final closed-loop system is almost-globally asymptotically stable. In this paper, we enlarge the family of controllers based on these ideas, giving several examples of control laws that fulfil the control objective and we broaden also the pumping-damping strategies with the new concept of circumscribing ovals.

The paper is organized as follows. Section 2 presents a family of suitable target energy functions for the pendulum, as well as the pumping-damping idea. In Section 3, several examples of controllers belonging to the family are presented. In Section 4 a stability criterion which generalices the one in (Åström *et al.*, 2005) is introduced. The paper closes with a Section of conclusions.

2. ENERGY SHAPING AND PUMPING-DAMPING

The normalized model of the pendulum system is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - u \cos x_1, \end{aligned} \tag{1}$$

being x_1 the angular position of the pendulum (the origin at the upright position) and x_2 the velocity, so it is defined on the manifold $\mathcal{S} \times R$.

To design a controller for the swing up problem the potential energy shaping method is used. For the moment we will focus in the choice of the energy function and, thus, a conservative target system will be chosen, leaving for later damping-pumping addition. For this, consider the desired system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -V_d'(x_1), \end{aligned} \tag{2}$$

which is a Hamiltonian system with Hamiltonian function

$$H_d(x_1, x_2) = V_d(x_1) + \frac{x_2^2}{2}, \tag{3}$$

where V_d should have a single minimum at the desired upright position. Thus, we should impose that $V_d'(0) = 0$, $V_d''(0) > 0$, and for symmetry reasons, $V_d(x_1) = V_d(-x_1)$.

To solve the matching problem of (1) and (2) a good choice of V_d', in order to avoid the division by $\cos x_1$, is

$$V_d' = -\sin x_1 + \beta(x_1) \cos x_1, \tag{4}$$

and then, $u = \beta(x_1)$.

A family of functions V_d that fulfills these conditions for appropriate values of parameters a_i is given by

$$V_d = a_0 + \cos x_1 - a_2 \cos^2 x_1 - a_3 \cos^3 x_1 - \cdots, \tag{5}$$

which yields

$$V_d' = -\sin x_1 + \sin x_1 \cos x_1 (2a_2 + 3a_3 \cos x_1 + \ldots), \tag{6}$$

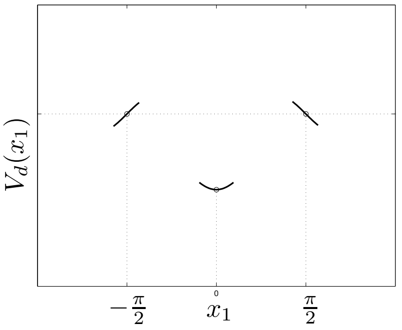

Fig. 1. Main properties shared by the V_d belonging to the family (5).

and therefore $\beta(x_1) = \sin x_1 (2a_2 + 3a_3 \cos x_1 + \ldots)$ to match expression (5) with (4). It should be noted that function H_d given by (3) and (5) is conservative for system (1) with $u = \beta(x_1)$.

The simplest case of this family is obtained by taking $a_0 = -1/4a$, $a_2 = a$ and $a_k = 0, \forall k > 2$, which leads to

$$V_d(x_1) = \cos x_1 - a \cos^2 x_1 - \frac{1}{4a}. \tag{7}$$

and then to the feedback law

$$u = u_{es} = 2a \sin x_1, \tag{8}$$

where the notation u_{es} has been introduced in order to point out that this is an energy-shaping control law. This controller is studied in (Åström *et al.*, 2005).

We consider now the problem of the choice of the energy variation strategy. Notice that a pure damping control law would not work. To notice this, consider the shape of any V_d belonging to family (5). It can be seen that it displays the traits shown in Fig. 1. This means the there is at least a maximum at $x_1 = x_1^0$ in the interval $(0, \pi/2)$ (resp. $(-\pi/2, 0)$). These maxima will give rise to saddles in the energy function (3). Out of the interval $(-x_1^0, x_1^0)$ there must exist at least one minima (see Fig. 1). With pure damping strategies, these extra minima give rise to undesired attraction basins, which we will call in the sequel "undesirable wells" in the energy landscape, because they preclude the global nature of the stability of the equilibrium at the upright position.

To overcome the difficulty with the undesirable wells we propose a strategy that consists in pumping energy to make the trajectories to leave them. This strategy is discussed in the following.

System (1) has as passive output $y = -x_2 \cos x_1$. This means that taking as input $u = -k_a y$ the system behaves in such a way that H_d decreases. However our goal is to increase H_d in the regions where pumping of energy is needed. Therefore we should modulate the sign of k_a according needs.

If we make $u = u_{es} + u_{pd}$ (for pumping-damping) in (1), the instantaneous energy variation is given by

$$\dot{H}_d = -x_2 u_{pd} \cos x_1. \tag{9}$$

We take $k_a = bF(x_1, x_2)$ where $b > 0$ and function F is negative in the region of the state space where we want to pump energy into the system and positive where we desire to damp the system. Then, $u_{pd} = bx_2 F(x_1, x_2) \cos x_1$, and therefore the control law is

$$u = \underbrace{\beta(x_1)}_{u_{es}} + \underbrace{bx_2 F(x_1, x_2) \cos x_1}_{u_{pd}}. \tag{10}$$

The first term of this controller can be interpreted as a nonlinear spring and we can therefore call it the "spring term". This term makes the pendulum to behave conservative. The second one is the pumping-damping term. In the next Section the problem of choosing appropriate F functions for different V_d is discussed.

3. DIFFERENT CHOICES OF THE ENERGY FUNCTION

In this Section three concrete cases for the energy function V_d are considered. All the cases are based on the same idea: first, an V_d belonging to the family (5) and with a minimum at the origin is chosen; then a control law is defined with the following structure:

$$u = u_{es} + bx_2 F(x_1, x_2) \cos x_1, \tag{11}$$

where u_{es} shapes the energy in order to obtain the corresponding

$$H_d(x_1, x_2) = V_d + \frac{x_2^2}{2}, \tag{12}$$

and $F(x_1, x_2) = 0$ is an appropriated oval that circumscribes about the boundaries of the undesirable wells. Parameter $b > 0$ defines the amount of energy that is pumped or damped.

3.1 A case with three wells

Consider the following potential energy function that belongs to the family (5)

$$V_{d_A}(x_1) = -\frac{\cos 3x_1}{3} - \frac{1}{3} = \cos x_1 - \frac{4}{3}\cos^3 x_1 - \frac{1}{3}, \tag{13}$$

which is shown in Fig. 2. It can be seen that it has three minima. The central one corresponds to the desired position and the other two are undesirable. The constant term in this function has been chosen so that the undesirable wells are bounded by $H_{d_A} = V_{d_A} + x_2^2/2 = 0$. These boundaries are shown in Fig. 3.

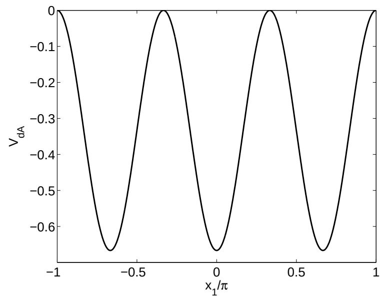

Fig. 2. Graph of $V_{d_A}(x_1)$.

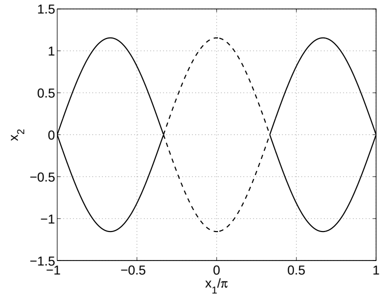

Fig. 3. Curve $\frac{x_2^2}{2} + V_{d_A}(x_1, x_2) = 0$. In solid the boundaries of the undesirable wells.

V_{d_A} of Eq. (13) yields

$$u_{es_A} = 4\cos x_1 \sin x_1 = 2\sin 2x_1, \tag{14}$$

that accomplishes the matching goal.

Note in Fig. 3 that neither of the undesirable wells contains the hanging position, which is an unstable equilibrium of saddle type. This fact induces a difference with respect to the case studied in (Åström *et al.*, 2005): Here the equilibrium $(x_1, x_2) = (\pi, 0)$ is a saddle even in absence of pumping-damping, while in (Åström *et al.*, 2005) this equilibrium is a center for $b = 0$ and unstable for $b \neq 0$.

In order to find a curve that circumscribes about the undesirable wells we choose a family of curves of the form

$$\frac{x_2^2}{2} + \alpha_0 + \alpha_1 \cos x_1 + \alpha_2 \cos^2 x_1 = 0. \tag{15}$$

Imposing that the curve passes through the points $(\pi/3, 0)$ and $(\pi, 0)$ and that the curve is tangent to $x_2^2/2 + V_{d_A} = 0$ we obtain:

$$F_A(x_1, x_2) = \frac{x_2^2}{2} - 1 + \cos x_1 + 2\cos^2 x_1 \tag{16}$$

The curve $F_A = 0$ is represented in Fig. 4. With respect to (Åström *et al.*, 2005), a term in $\cos^2 x_1$ has been added to shape appropriately the oval.

Therefore, we propose a controller such as (11) that now becomes

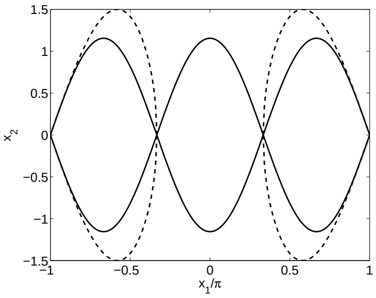

Fig. 4. Shape of $F_A = 0$ (dashed) circumscribing about the boundaries of the undesirable wells associated with the curve $x_2^2/2 + V_{d_A}(x_1) = 0$ (solid).

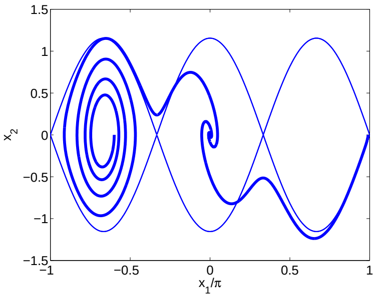

Fig. 5. Results of two simulations corresponding to controller (17) with $b = 0.4$.

$$u = 2a\sin 2x_1 + bx_2 F_A(x_1, x_2)\cos x_1, \qquad (17)$$

with F_A given by (16).

Simulations show the good performance of this controller (Fig. 5). If the initial condition is the hanging position at rest, this controller does not need energy injection (as does the one in (Åström *et al.*, 2005)) for leaving this hanging position.

Remark 1. Other choices of circumscribing curves are possible. For example (Fig. 6)

$$\tilde{F}_A(x_1, x_2) = \cos x_1 - \frac{1}{2} + \frac{2x_2^2}{3}, \qquad (18)$$

which corresponds to $\alpha_2 = 0$, and remembers the ovals used in (Åström *et al.*, 2005). Simulations show that this controller works satisfactorily. However, it will be seen below that in this case the stability criteria proposed in the current paper does not guaranty stability. This fact justifies the inclusion of the term $\cos^2 x_1$ in F.

3.2 A case with asymmetrical undesirable wells

Another interesting V_d (represented in Fig. 7) is obtained by the choice $a_0 = -\sqrt{2}/6$, $a_3 = 8/3$ and $a_{2k+1} = 0, \forall k \neq 0$ in (5), which gives

$$V_{d_B}(x_1) = \cos x_1 - \frac{8}{3}\cos^3 x_1 - \frac{\sqrt{2}}{6}. \qquad (19)$$

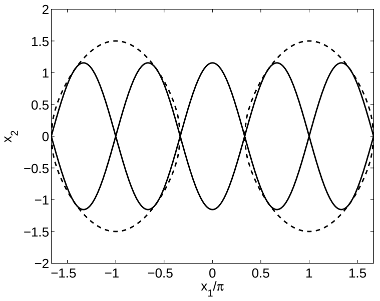

Fig. 6. Curves $x_2^2/2 + V_{d_A}(x_1, x_2) = 0$ (solid) and $\tilde{F}_A = 0$ (dashed).

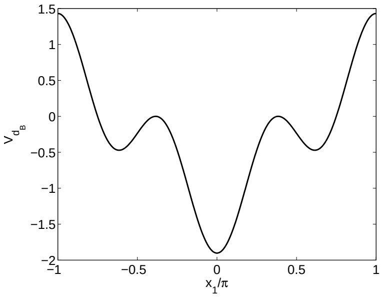

Fig. 7. Graph of $V_{d_B}(x_1)$.

As before, the term a_0 has been chosen so that the undesirable wells are bounded by $V_{d_B} = 0$. In this last case the matching yields

$$u_{es_B} = 4\sin 2x_1. \qquad (20)$$

The shape of V_{d_B} is shown in Fig. 7. It displays two undesirable wells that are asymmetrical, in the sense that the maximum values at both sides of the undesirable wells are different.

The asymmetry of the undesirable wells leads to the fact that, in absence of pumping and damping, their boundaries are no more heteroclinic orbits but homoclinic ones (Fig. 8). Figure 8 shows that the closed curve defined by

$$\frac{x_2^2}{2} - \frac{4+\sqrt{2}}{6} + \frac{1+2\sqrt{2}}{3}\cos x_1 + \frac{2\sqrt{2}+8}{3}\cos^2 x_1 = 0, (21)$$

circumscribes about the two undesirable wells. It is worthy to note that in this case the circumscribing curve is split in two, one for each well.

As before, simulations show the good performance of this controller (Fig. 9).

3.3 A more sophisticated case

Many other V_d belonging to the same family can be conceived. Recalling (5), this expression can be rewritten as

$$V_d = a_0 + \cos x_1 - \cos^2 x_1(a_2 + a_3\cos x_1 + \cdots,)$$

therefore

$$V_d = a_0 + \cos x_1 - \cos^2 x_1 f(\cos x_1)$$

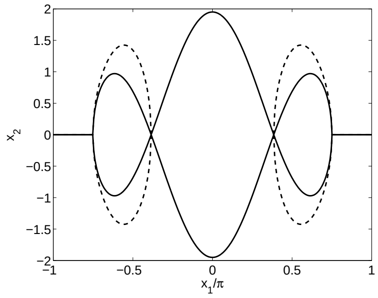

Fig. 8. Curves $x_2^2/2 + V_{d_B}(x_1, x_2) = 0$ (solid) and $F_B = 0$ (dashed).

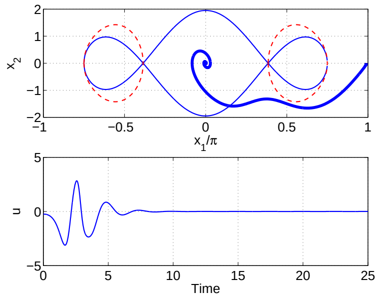

Fig. 9. Results of a simulation corresponding to the case (20)–(21) with $b = 0.4$.

where $f(y)$ has to be chosen such that V_d has the appropriate shape (a minimum at the desired upright position) and it has a series expansion of the form

$$f(y) = b_0 + b_1 y + b_2 y^2 + \cdots$$

For instance, $f(y) = \exp(y)(1 + y)$ yields

$$V_{d_C} = \cos x_1 - \cos^2 x_1 \exp(\cos x_1)(1 + \cos x_1) - 0.1497, \quad (22)$$

which is shown in Fig. 10. With V_{d_C} and applying the same procedure as before

$$u = \sin x_1 \exp(\cos x_1)(2 + 4\cos x_1 + \cos x_1^2) + b x_2 F_C(x_1, x_2) \cos x_1, \quad (23)$$

where a reasonable oval circumscribing about the only undesirable well is given by

$$F_C(x_1, x_2) = 2x_2^2 + 4\cos x_1 - 1. \quad (24)$$

Figure 12 shows the good results for two simulations of the system with the resultant control law.

4. A STABILITY CRITERION

The stability criterion presented in (Åström *et al.*, 2005) for the concrete case studied there, is generalized here for the whole family of energy functions (5) considered in the current paper. For each $V_d(x_1, x_2)$ having a minimum at the origin, the corresponding $F(x_1, x_2)$ is chosen in such a way that it circumscribes about the undesirable wells. Once function F is chosen, the following functions can be defined:

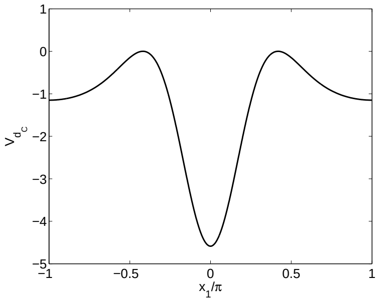

Fig. 10. Graph of $V_{d_C}(x_1)$.

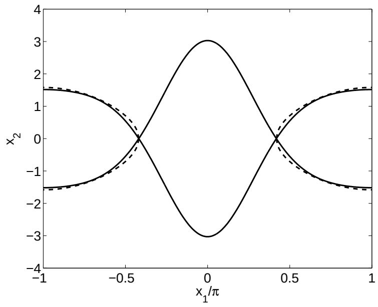

Fig. 11. Curves $x_2^2/2 + V_{d_C}(x_1)=0$ (solid) and $F_C = 0$ (dashed).

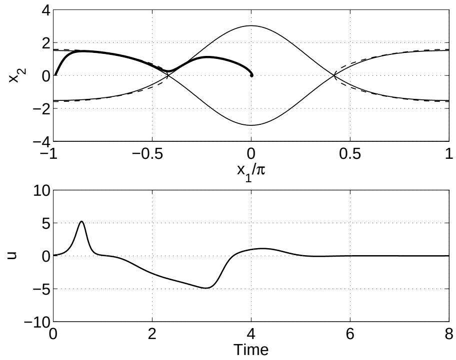

Fig. 12. Example of trajectory (top) and control signal (bottom) showing the good behavior of controller (23-24) with $b = 1.75$.

- $\varphi_H(x_1)$ (with $x_1 \in [0, \pi]$) as the x_2 coordinate of the upper curve defined by $H(x_1, x_2) = V(x_1) + x_2^2/2 = 0$. For the values of x_1 where no value of x_2 satisfies this last expression, take $\varphi_H(x_1) = 0$.
- $\varphi_F(x_1)$ as the x_2 coordinate of the upper curve defined by $F(x_1, x_2) = 0$. This curve will only be defined in some set Ω. Let $\bar{\Omega}$ be the complement set of Ω in $[0, \pi]$, that is $\bar{\Omega} = [0, \pi] \backslash \Omega$.
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$$\Phi = \int_{\bar{\Omega}} \varphi_H(x_1) \cos^2(x_1) F(x_1, \varphi_H(x_1)) dx_1 + \int_{\Omega} \varphi_F(x_1) \cos^2(x_1) F(x_1, \varphi_H(x_1)) dx_1 \quad (25)$$

Theorem 1. Consider system (1) with control law (11) and F defined in such a way that $F = 0$ circumscribes about the boundaries of the undesirable wells. The origin is almost-globally asymptotically stable if $\Phi > 0$.

Proof : Only a sketch of the proof is given here. Due to the pumping mechanism inside the ovals, it is easy to see that trajectories go out of the undesirable wells. Consider a trajectory once out of any undesirable well. It can reach the desired well or turn around the manifold $\mathcal{S} \times R^1$. In the first case, asymptotic stability is guaranteed by Lyapunov theory. In the second case, we can compute the energy balance through a 2π turn. From Eq. (9) the energy change along such a turn is given by

$$\Delta H_d(x_1, x_2) = -\int_{-\pi}^{\pi} b x_2 \cos^2 x_1 F(x_1, x_2) dx_1,$$

where ΔH_d is the net energy loss along a 2π cycle of x_1. It can be shown that $\Delta H_d > \Phi$, because Φ, according to its definition in (25), gives a conservative measure of the bounds of the net energy loss during a 2π cycle (for details, see (Åström *et al.*, 2005)). This measure is conservative in the sense that energy injection is overestimated, while damping is underestimated. In consequence $\Phi > 0$ means that the dissipation is higher than energy injection, and so the trajectory in the state space shrinks until it reaches the central well where stability is guaranteed by Lyapunov theory.

Table 1 shows the corresponding values of Φ for the examples given above. As was said before, stability is not guaranteed for the third case in this table, since for this case $\Phi < 0$, in spite of the fact that the system seems to be stable by simulation. This fact shows the conservativeness of the criterion. In any case, the corresponding controller was included just for illustrative purposes since it is clearly outperformed by the ones corresponding to line 2 in Table 1. For the rest of the proposed controllers Theorem 1 guarantees stability. An open problem is the performance comparison of the different choices for potential energy functions as well as for the circumscribing ovals.

Table 1. Values of Φ for the examples of Sect. 3

$V_d(x_1)$	$F(x_1, x_2)$	Φ
V_d in (Åström *et al.*, 2005) with $a = 1$	F in (Åström *et al.*, 2005)	0.0598
V_{d_A} Eq. (13)	F_A Eq. (16)	1.9846
V_{d_A} Eq. (13)	$\tilde{F}_A$ Eq. (18)	−0.4481
V_{d_B} Eq. (19)	F_B Eq. (21)	7.0034
V_{d_C} Eq. (22)	F_C Eq. (24)	28.3871

5. CONCLUSIONS

In this paper a further elaboration of the approach to the swing up problem of the inverted pendulum introduced in (Åström *et al.*, 2005) is presented. Te approach is based in a two step procedure. First a feedback law is applied such that the desired equilibria becomes stable although not asymptotically stable. However, this introduces more equilibria in the closed loop system. Then, the damping term is added to introduce damping around the desired equilibrium to change it from being stable to being asymptotically stable; and moreover, negative damping (energy pumping) is introduced in the region around the undesirable equilibria to make them unstable. In this way a single controller is obtained that swings up the pendulum and stabilizes it from almost all initial conditions (positions and velocities). In the present paper the results of the previous one are extended to a wider class of energy functions. Furthermore, a new concept of circumscribing ovals is introduced that considerably deepens the approach. For these energy functions and circumscribing ovals a stability criterion is stated that generalices the one of (Åström *et al.*, 2005). Four different controllers based on this approach are included in the paper.

Acknowledgments

The Spanish authors have been supported under the MCyT-FEDER grant DPI2003-00429.

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[1] We are considering only the case where, in the definition of Φ, $\Omega = [0, \pi]$ and consequently $\hat{\Omega} = \emptyset$. The case where $\hat{\Omega} \neq \emptyset$ is more involved.