ANALYSIS OF THE CONTROL SATURATION IN PUMPING-DAMPING STRATEGIES FOR THE INVERTED PENDULUM

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Abstract: In this paper the control law presented in (Åström *et al.*, 2005) is modified in order to deal with saturation in the control signal. It is shown that the problem can be solved without leaving the Hamiltonian shaping approach. An almost-global stability result provides values for the tuning parameters.

Keywords: Pendulum, Shaping Hamiltonians, Swing up, Energy management, control saturation.

1. INTRODUCTION

The inverted pendulum presents two main problems: swinging up the pendulum to the upright position (Åström and Furuta, 2000; Gordillo *et al.*, 2003) and stabilizing it in this position once it is reached. These problems have traditionally been treated as two separate ones. The first one is usually solved by energy considerations (Åström and Furuta, 2000). However, this kind of controller leads to the stabilization of an homoclinic orbit (Lozano *et al.*, 2000). In this way, the system will eventually approach the desired equilibrium but without achieving local stability since, due to small disturbances, the system will go away from this point. One explanation of this behavior is that the desired point is a saddle point and the (attractive) homoclinic orbit is its stable manifold. On the other hand there are laws that make this point asymptotically stable (Bloch *et al.*, 1999) but the domain of attraction is limited and never reaches the horizontal plane.

Then, the full problem is usually solved by switching between different laws: first, a law that performs the swing-up is used and, once the pendulum is near the vertical position (or at least above the horizontal), the controller switches to a local law (Wiklund *et al.*, 1993; Shiriaev *et al.*, 2001). To stabilize the pendulum when an appropriate neighborhood of the upper position is reached linear methods are good enough.

In (Åström *et al.*, 2005) an energy shaping control law was designed in such a way that: 1) the closed-loop energy presents a minimum at the desired position; and 2) the energy shaping controller is globally defined. Since the chosen target energy has other minima different than the desired equilibrium, a combination of energy dissipation (damping) and injection (pumping) is needed in order to globally stabilize the origin. To that end an oval closed curve circumscribing about the region where pumping is needed was introduced. The resultant law is smooth, no commutations are needed, and the origin of the final closed-loop system is almost-globally asymptotically stable.

In this paper, we deal with the problems associated with the presence of control signal saturation. The motivation comes from the fact that, unlike other swing-up laws such as (Åström and Furuta, 2000), the law in (Åström *et al.*, 2005) can not be designed arbitrarily small. We can consider that the cost for achieving a unique control law that solves both the swing-up and the stabilization problems is the necessity of considerable control actions. The main contributions of the paper are the modification of the original control law for the case of control input saturation as well as the extension of the stability proof given in (Åström *et al.*, 2005) giving conditions on the control law tuning parameters for assuring stability.

The rest of the paper is organized as follows. In Sect. 2 the control law presented in (Åström *et al.*, 2005) is recalled. An explanation of the reason why the control signal can not be arbitrarily small in energy shaping control laws is given. In Sect. 3 necessary modifications are introduced in the original control law in order to cope with saturation. Section 4 is devoted to the stability analysis by using a new Hamiltonian function, which is different from the one in the case of no saturation. Conclusions are presented in Sect. 5.

2. STATEMENT OF THE PROBLEM

The model of the pendulum, when the control action is the acceleration of the pivot, is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - u \cos x_1, \end{aligned} \qquad (1)$$

where x_1 is the angular position of the pendulum with the origin at the upright position and x_2 is the velocity of the pendulum. This system is defined on a cylindrical state space $S \times R$. Notice that for $u = 0$ the system is Hamiltonian with a Hamiltonian function $H = \cos x_1 + x_2^2/2$.

In (Åström *et al.*, 2005) a smooth control law that was able to swing up and stabilize the pendulum was designed. The law $u(x_1, x_2)$ has two parts $u = u_{es} + u_{pd}$. The first term, u_{es} –which stands for energy shaping– shapes the Hamiltonian so that:

(1) the resultant Hamiltonian has a minimum at the origin;
(2) the resultant energy shaping control law is globally defined (e.g. the term $\cos x_1$ must not appear in its denominator).

It can be shown that in order to fulfill these two requirements, the desired Hamiltonian necessarily has other undesirable minima (Aracil *et al.*, 2006). For this, the second term u_{pd} is not the usual damping term but a pumping-damping term in the sense that energy is injected in some regions which include the undesirable minima, while the system is damped elsewhere.

In absence of friction, the term u_{pd} can be made arbitrarily small by decreasing arbitrarily a positive tuning parameter. The only consequence is that the transient behavior is slower. On the contrary, upper bounds can not be arbitrarily imposed on the energy shaping term absolute value. The reason is given in the following. Since the goal of this term is to introduce changes in the Hamiltonian, one could think that looking for a desired Hamiltonian function, $H_d(x_1, x_2) = V_d(x_1) + x_2^2/2$ (where V_d stands for the desired potential energy) close to the original one would result in a small energy-shaping term. One possibility is shown Fig. 1 where the original potential energy ($V(x_1) = \cos x_1$) is plotted with a possible choice of V_d. The choice plotted in this figure is very close to the original except around the origin where we want to change the maximum of V into a minimum. Let

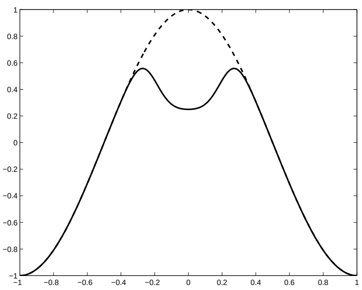

Fig. 1. Comparison of potential energies: original $V(x_1) = \cos x_1$ (dashed) and a possible choice of V_d (solid).

us analyze the resultant energy shaping term. The Hamiltonian system associated with the desired Hamiltonian is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -V_d'(x_1), \end{aligned} \qquad (2)$$

and, thus, the energy shaping control law that matches (1) and (2) is

$$u_{es} = \frac{V_d'(x_1) + \sin x_1}{\cos x_1} = \frac{V_d'(x_1) - V'(x_1)}{\cos x_1}. \qquad (3)$$

Of course, the choice of $V_d(x_1)$ has to be such that the term $\cos x_1$ is cancelled in (3). Nevertheless, what it is important to note here is that the magnitude of the energy-shaping term depends on $V_d'(x_1) - V'(x_1)$, that is, on the derivative of the potential energy increment instead of the increment itself. This fact brings an important consequence, which can be deduced from Fig. 1: since we want to change the maximum of V into a minimum of V_d, the slopes of the two curves in the figure has to differ in a significative amount at some points. This means that $V_d' - V'$ must reach significative values, which implies that the energy shaping magnitude has to be significative, at least at some points, irrespective of the choice of $V_d(x_1)$.

2.1 A pumping-damping control law

The control law obtained in (Åström *et al.*, 2005) is based on the choice $V_d(x_1) = \cos x_1 - a\cos^2 x_1$ with $a > 0.5$, which gives $u_{es} = 2a\sin x_1$. The pumping-damping term is given by

$$u_{pd} = bx_2 F(x_1, x_2)\cos x_1 \quad (4)$$

$$F(x_1, x_2) = \frac{x_2^2}{2} + \frac{2a+1}{4a}(2a\cos x_1 - 1), \quad (5)$$

with $b > 0$. Function F determines the sign of the energy variation: $F > 0$ means damping, while $F < 0$ implies energy injection. Function F is chosen so that the curve $F(x_1, x_2) = 0$ circumscribes about the undesirable energy wells (see Fig. 2 taken from (Åström *et al.*, 2005)).

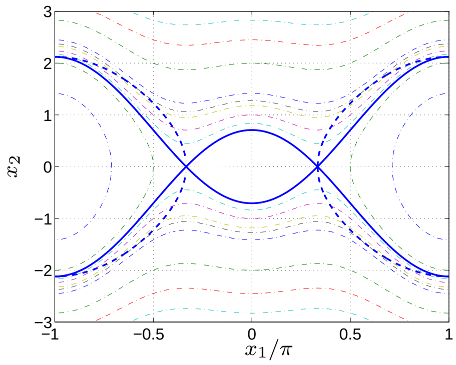

Fig. 2. Curves $H_d(x_1, x_2) = 0$ (solid), $F(x_1, x_2) = 0$ (dashed) and some level curves for function $H(x_1, x_2)$ (dot-dashed).

In (Åström *et al.*, 2005) almost-global stability was proved for the resultant system. The proof was based on obtaining a conservative bound for the energy balance. Nevertheless, the possible saturation of the control signal was not taken into account. In this paper we deal with the constrain $|u| < \bar{u}$. This saturation may have important impact in the behavior of the system when $\bar{u} < 2a$ since the required value for u_{es} may reach this value. Not only the stability proof is not longer valid for this case but this is an actual problem since it can be seen in simulations that the law does not work for this case.

3. MODIFICATIONS OF THE CONTROL LAW

This section is devoted to the modification of the control law for the case potential energy chosen in (Åström *et al.*, 2005) in order to cope with saturations in the control signal.

3.1 Energy shaping in the presence of saturation

First of all, consider system (1) with $u = u_{es} = 2a\sin x_1$ and, thus, without considering the pumping-damping term. Assume that the controller saturates at $u = \hat{u}$, i.e., $|u| \le \hat{u}$ (for a start, $\hat{u}$ may be considered equal to $\bar{u}$, but it will redefined below) . Then, we have for the closed loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - \cos x_1 \mathrm{sat}_{\hat{u}}(2a\sin x_1), \end{aligned} \quad (6)$$

where $\mathrm{sat}_{\hat{u}}$ denotes the saturation at $\bar{u}$ $\mathrm{sat}_{\hat{u}}(\cdot) = \mathrm{sgn}(\cdot)\min\{|\cdot|, \hat{u}\}$. Define in the plane (x_1, x_2) the region R_1 as the one such that $|\sin x_1| \le \hat{u}/(2a)$ (this region consists of vertical bands around the origin and around $(\pm\pi, 0)$, see Fig. 3). In this region the system behaves as without saturation, with $H_d = x_2^2/2 + \cos x_1 - a\cos^2 x_1$. For $\sin x_1 > \hat{u}/(2a)$ we define region R_2, in which the system behavior is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - \hat{u}\cos x_1. \end{aligned} \quad (7)$$

Notice that system (7) is Hamiltonian, with Hamiltonian function given by

$$H_2(x_1, x_2) = x_2^2/2 + \cos x_1 + \hat{u}\sin x_1. \quad (8)$$

This means that H_2 is constant for dynamics (7), and consequently H_2 is constant when the state of the system is in region R_2.

Similarly we define a third region R_3 for $\sin x_1 < -\hat{u}/(2a)$ and where system behavior is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 + \hat{u}\cos x_1. \end{aligned} \quad (9)$$

System (9) is also Hamiltonian. Its Hamiltonian function is now given by $H_3(x_1, x_2) = x_2^2/2 + \cos x_1 - \hat{u}\sin x_1$. Consequently, H_3 is constant when the state of the system is in region R_3.

For region R_1, as we are considering the case without the damping-pumping term, function H_d is also constant. Nevertheless, it is convenient to choose a new reference for the potential energy in order to achieve continuity at the boundaries of the regions. Define

$$H_1 = \cos x_1 - a\cos^2 x_1 + \frac{x_2^2}{2} + \frac{\hat{u}^2}{4a} + a \quad (10)$$

Notice that, with this choice, we have achieved continuity of functions H_1 and H_2 at the boundary between regions R_1 and R_2, i.e, at $x_1 =$

arcsin $\hat{u}/(2a)$ (notice also that, in fact, there are two of such boundaries in $x_1 \in [-\pi, \pi]$, since function arcsin gives two values):

$$H_1\left(\arcsin\frac{\hat{u}}{2a}, x_2\right) = H_2\left(\arcsin\frac{\hat{u}}{2a}, x_2\right) \quad \forall x_2.$$

In the same way, functions H_1 and H_3 have the same value at the boundary between regions R_1 and R_3, i.e., at $x_1 = -\arcsin\hat{u}/(2a)$:

$$H_1\left(-\arcsin\frac{\hat{u}}{2a}, x_2\right) = H_3\left(-\arcsin\frac{\hat{u}}{2a}, x_2\right) \quad \forall x_2.$$

In this way, defining the function

$$H_{\text{sat}} = \begin{cases} H_1 \text{ if } x \in \mathrm{R}_1, \\ H_2 \text{ if } x \in \mathrm{R}_2, \\ H_3 \text{ if } x \in R_3, \end{cases} \qquad (11)$$

we can say that function H_{sat} is continuous. It can be seen that the derivatives of H_{sat} are also continuous. Figure 3 shows some level curves for this function. Therefore, the qualitative descrip-

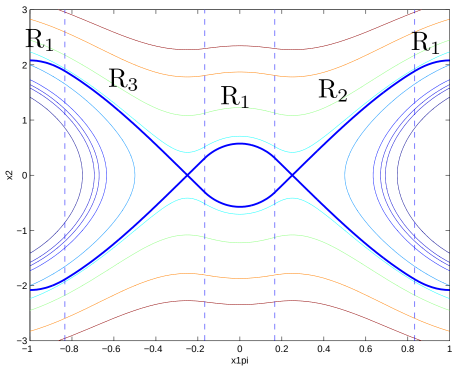

Fig. 3. Level curves for function H_{sat}, $a = 1, \hat{u} = 1$.

tion of the system with saturation (Fig. 3) is very similar to one corresponding to the unsaturated case (Fig. 2): both systems are Hamiltonian; nevertheless their Hamiltonians are different: H_d given by $H_d = H_1$ in the unsaturated case and $H = H_{\text{sat}}$ given by (11) in the case of saturation.

Before dealing with the problem of adding pumping and damping, it is important to compute the level set $H_{\text{sat}} = H^*$ corresponding to the saddle points of this system $(x_1, x_2) \triangleq (x_1^0, 0)$ since this level defines the undesirable wells. Two cases can occur:

(1) The saddles points lie in region R_1. It is straightforward to see that this case happens when $\hat{u} \geq \sqrt{4a^2 - 1}$. In this case, the level set $H_{\text{sat}} = H^*$ has to be computed using expression (10) giving rise to

$$H^* = \frac{1 + \hat{u}^2}{4a} + a.$$

(2) The saddles points lie out of region R_1. This case happens when $\hat{u} < \sqrt{4a^2 - 1}$. In this case, the level set $H_{\text{sat}} = H^*$ has to be computed using expression (8) giving rise to $H^* = \sqrt{1 + \hat{u}^2}$.

3.2 Pumping and damping

The introduction of the pumping-damping term in a similar way to (Åström *et al.*, 2005) has to be done carefully since the pumping-damping term can distort the boundary between the zones making that Hamiltonian H_{sat} is not longer continuous and, thus, invalidating stability analysis similar to the one of (Åström *et al.*, 2005). An easy way to avoid this problem is to introduce u in the following way:

$$u = \text{sat}_{\delta\bar{u}}(u_{es}) + \text{sat}_{(1-\delta)\bar{u}}(u_{pd}), \qquad (12)$$

with $0 < \delta < 1$. In this way the energy shaping term is saturated at $\hat{u} = \delta\bar{u}$ (this is the reason why a symbol different from $\bar{u}$ was used for the saturation of u_{es} in the previous section) and the pumping-damping term does not interfere with it. With this choice, the analysis of Sect. 3.1 is still valid (of course, Hamiltonian (11) is no longer constant but it is modulated by u_{pd}).

As in (Åström *et al.*, 2005), the pumping-damping term is chosen of the form (4) but function F has to be redefined since it must circumscribe about the undesirable wells. We choose function F of the form $F(x_1, x_2) = \alpha_2 x_2^2 + \alpha_1 \cos x_1 - 1$. We will impose that the curve $F = 0$ intersect with the curve $H_{\text{sat}} = H^*$ at the saddle points and at $x_1 = \pi$. The two previous cases have to be separated:

(1) For $\hat{u} \geq \sqrt{4a^2 - 1}$, as the curves $F = 0$ and $H_{\text{sat}} = H^*$ intersect in the non-saturated region (R_1), this cases coincides with the one of (Åström *et al.*, 2005):

$$\alpha_1 = 2a$$
$$\alpha_2 = \frac{2a}{2a + 1}.$$

(2) For $\hat{u} < \sqrt{4a^2 - 1}$, the point in the curve $H_{\text{sat}} = H^*$ corresponding to $x_1 = \pi$ has a value of x_2 equal to

$$\frac{\sqrt{4a^2 - \hat{u}^2 a + 4a^2\sqrt{1 + \hat{u}^2}}}{\sqrt{2}a},$$

and, after some computations we obtain:

$$\alpha_1 = \sqrt{1 + \hat{u}^2}$$
$$\alpha_2 = \frac{1 + \alpha_1}{2(1 + \alpha_1) - \hat{u}^2/(4a)}$$

Figure 4 represents the curves $H_{\text{sat}} = H^*$ and $F = 0$ for examples of both cases. In this figure, regions A, B_1, B_2 and C are defined as the regions limited by curves $F = 0$ and/or $H_{\text{sat}} = H^*$.

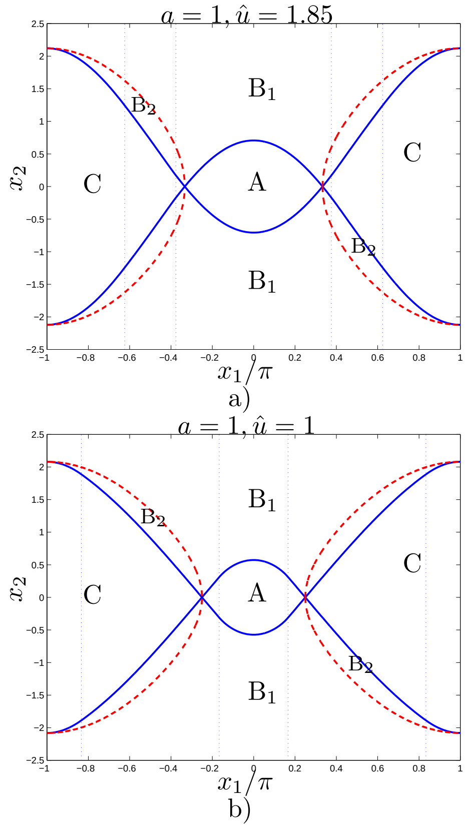

Fig. 4. Example of curves $H_{\text{sat}} = H^*$ and $F = 0$ for a) $\hat{u} > \sqrt{4a^2 - 1}$ and b) $\hat{u} < \sqrt{4a^2 - 1}$. The vertical lines limit the saturation zone.

4. STABILITY ANALYSIS

Function H_{sat} can serve as a basis for the generalization of the analysis carried out in (Åström *et al.*, 2005) and the stability proof for the saturation case can be easily obtained. Here, only the main points are presented. The closed-loop system is

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \sin x_1 - \text{sat}_{\delta\bar{u}}(u_{es}) + \text{sat}_{(1-\delta)\bar{u}}(u_{pd}),\end{aligned}$$

with $u_{es} = 2a\sin x_1$ and $u_{pd} = bx_2F(x_1, x_2)\cos x_1$. We have chosen function H_{sat} such that, for $u_{pd} = 0$, $\dot{H}_{\text{sat}} = 0$. Therefore,

$$\begin{aligned}\dot{H}_{\text{sat}} &= -x_2\cos x_1 \text{sat}_{(1-\delta)\bar{u}}(u_{pd})\\ &= -x_2\cos x_1 \text{sat}_{(1-\delta)\bar{u}}(bx_2F(x_1, x_2)\cos x_1).\end{aligned}$$

Notice that, if $b > 0$, the sign of $\dot{H}_{\text{sat}}$ is determined by the sign of F. In this way, H_{sat} is increasing inside the undesirable wells (region C) and, thus, trajectories leave this region (except trajectories starting at rest at the downward position that is an unstable equilibrium point). Furthermore, H_{sat} is decreasing in region A and, thus, H_{sat} is a Lyapunov function in this domain. The other regions have to be analyzed carefully since H_{sat} is not always decreasing. Instead of using standard Lyapunov arguments, the net energy balance along full revolutions of the pendulum will be estimated. For this, notice that

$$D_{x_1}H_{\text{sat}} = \frac{dH_{\text{sat}}}{dx_1} = \frac{\dot{H}_{\text{sat}}}{\dot{x}_1} = -u_{pd}\cos x_1$$

that is,

$$-D_{x_1}H_{\text{sat}} = \text{sat}_{(1-\delta)\bar{u}}(bx_2F(x_1, x_2)\cos x_1)\cos x_1.$$

A lower bound for function $-D_{x_1}H_{\text{sat}}$ in regions B_1 and B_2 can be found in the following way:

- For points in region B_1 with x_1 between the saddle points ($|x_1| < x_1^0$), since $F > 0$, we have

$$-D_{x_1}H_{\text{sat}} \geq \text{sat}_{(1-\delta)\bar{u}}(b\varphi_H(x_1)F(x_1, \varphi_H(x_1))\cos x_1)\cos x_1,$$

  where $\varphi_H(x_1)$ represents the x_2 coordinate of the upper curve defined by $H_{\text{sat}}(x_1, x_2) = H^*$. It can be obtained from Eq. (11).
- For points still in region B_1 but with $|x_1| > x_1^0$ we have

$$D_{x_1}H_{\text{sat}} \geq \text{sat}_{(1-\delta)\bar{u}}(b\varphi_F(x_1)F(x_1, \varphi_H(x_1))\cos x_1)\cos x_1,$$

  where $\varphi_F(x_1)$ represents the x_2 coordinate of the upper curve defined by $F = 0$, i.e.

$$\varphi_F(x_1) = \sqrt{\frac{1 - \alpha_1\cos x_1}{\alpha_2}}.$$

- For points in region B_2 since $F < 0$ we have

$$D_{x_1}H_{\text{sat}} \geq \text{sat}_{(1-\delta)\bar{u}}(b\varphi_F(x_1)F(x_1, \varphi_H(x_1))\cos x_1)\cos x_1.$$

Therefore,

$$\begin{aligned}-\Delta H_{\text{sat}} &\triangleq -\int_0^{\pi} D_{x_1}H_{\text{sat}}dx_1\\ &\geq \int_0^{x_1^0} \text{sat}_{(1-\delta)\bar{u}}(b\varphi_H(x_1)F(x_1, \varphi_H(x_1))\cos x_1)\cos x_1 dx_1\\ &+ \int_{x_1^0}^{\pi} \text{sat}_{(1-\delta)\bar{u}}(b\varphi_F(x_1)F(x_1, \varphi_H(x_1))\cos x_1)\cos x_1 dx_1\\ &\triangleq \Phi(a, b, \delta, \bar{u})\end{aligned}$$

It can be seen that stability is guaranteed if $\Delta H_{\text{sat}} \leq 0$ and this condition is fulfilled if $\Phi(a, b, \delta, \bar{u}) > 0$. Therefore, we can state the following theorem

Theorem 1. Consider system (1) with control law (12) with u_{es} and u_{pd} defined above. The origin is almost-globally asymptotically stable for any $a > 0.5$ such that $\Phi(a, b, \delta, \bar{u}) > 0$.

The applicability of this result is more involved than the one of the unsaturated case since, the corresponding function Φ for the last case only depends on a. Nevertheless, the previous theorem is still useful. For a given pendulum, parameter $\bar{u}$ has a physical value. The control designer has to look for values of parameters $a > 0.5, b > 0$ and $0 < \delta < 1$ such that $\Phi(a, b, \delta, \bar{u}) > 0$. For example, parameter b may be fixed and a graph such as the one of Fig. 5 can be obtained. In this figure, which corresponds to $\bar{u} = 2.5, b = 0.1$, the marks represent the values of a and δ that give a positive $\Phi(a, b, \delta, \bar{u})$ and, thus, give rise to an almost-globally stable system.

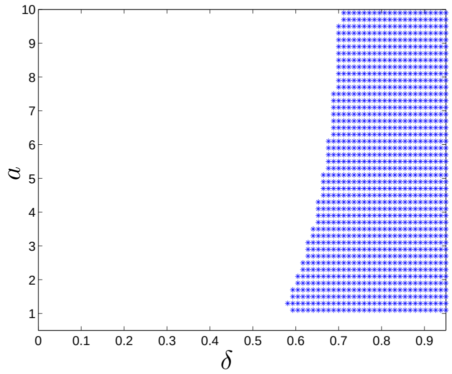

Fig. 5. Values of a and δ for which $\Phi(a, b, \delta, \bar{u}) > 0$. Graph for $b = 0.1$ and $\bar{u} = 2.5$.

Finally, we present the results of two simulations in order to show the usefulness of the modifications proposed. In the first simulation, the original control law presented in (Åström *et al.*, 2005) is simulated with saturation in the control action ($a = 10, b = 0.1, \bar{u} = 2.5$). It is shown that the controller does not work properly. In the second simulation the new controller with $\delta = 0.8$ is used showing a good behavior according to Fig. 6.

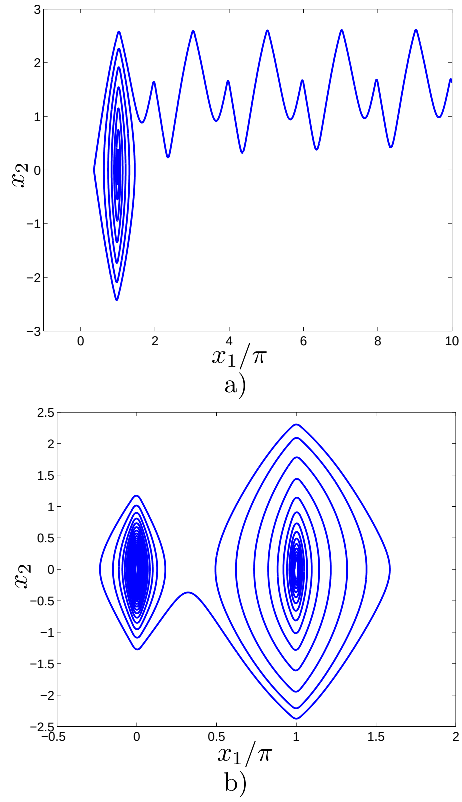

Fig. 6. Results of two simulations with $a = 10, b = 0.1$, and $\bar{u} = 2.5$: a) with the original controller (Åström *et al.*, 2005); b) with the new controller with $\delta = 0.8$.

5. CONCLUSIONS

In this paper, the control law for the inverted pendulum presented in (Åström *et al.*, 2005) has been extended to the case of saturation in the control action. First, it has been shown that, for this system, energy shaping control laws cannot be made arbitrarily small and, thus, problems related to control saturation may appear. Modifications to the original control law have been presented in order to cope with the saturation. These modifications do not leave the energy-shaping approach. An almost-global stability theorem has been stated.

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