On the PDEs arising in IDA-PBC

J.Á. Acosta and A. Astolfi

Abstract—The main stumbling block of most nonlinear control methods is the necessity to solve nonlinear Partial Differential Equations. In this paper we propose a methodology to replace the PDEs of Interconnection and Damping Assignment Passivity-Based Control with algebraic inequalities. The idea relies on the construction of an approximating integral and a dynamic extension as an alternative to solve the PDEs. Some connections with linear control theory and nonlinear cascade control techniques are also provided.

I. INTRODUCTION

Stabilization of nonlinear systems shaping their energy function, but preserving the systems structure, has attracted the attention of control researchers for several years. Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) is a generic name to define a controller design methodology based on the passivity property. In fact, Passivity-Based Control (PBC) is a controller design methodology [6] which achieves stabilization by rendering the system passive with respect to a desired storage function (energy) which has a minimum at the desired equilibrium point. PBC designs may be classified into two groups. A first group where a storage function to be assigned is selected and then the controller that renders the storage function non-increasing (i.e. a Lyapunov-like approach) is designed. In the second group, the desired structure of the closed-loop system is selected instead of fixing the closed-loop storage function. In the latter case, once the closed-loop structure has been selected the control problem is then to characterize all assignable storage functions (energy) compatible with this structure. This characterization is given in terms of the solution of Partial Differential Equations (PDEs). The most notable examples of this approach arise in the control of mechanical systems. While fully-actuated mechanical systems allow an arbitrary shaping of the energy by means of feedback, and therefore stabilization to any desired equilibrium, this is in general not possible for underactuated systems. In case of underactuated mechanical systems there are two main approaches: the method of Controlled Lagrangians [2], [3] and IDA-PBC [7], [8]. In both methodologies stabilization (of a desired equilibrium) is achieved identifying the class of systems, Lagrangian or Hamiltonian, respectively, that can possibly be obtained via feedback. The conditions under which such a feedback law exists are called matching conditions, and consist of a set of nonlinear PDEs. When these PDEs can be solved the original control system and the target dynamic system are said to match. In IDA-PBC the PDEs that we have to solve are parameterized by matrices that, roughly speaking, can be simply viewed as degrees-of-freedom to enforce the required passivity property and dissipation.

In this paper a methodology to replace the PDEs by algebraic inequalities is proposed. The idea relies on the construction of an approximating integral and a dynamic extension as an alternative to solve the PDEs. The approximating integral was proposed in [4] in the context of nonlinear observers. The approach is applied to IDA-PBC methodology and some connections with linear systems control theory and nonlinear cascade control techniques are discussed.

The outline of the paper is as follows. In section II the main idea and procedure of the IDA-PBC control methodology is described; the main theoretical result is given in Section III; in Sections IV and V the applicability of the method to systems in lower and upper triangular form, respectively, is discussed; and finally a conclusion section is given.

II. BACKGROUND

In [8] IDA-PBC has been introduced as a procedure to control physical systems described by a generalized Hamiltonian structure (see also [11]). Consider a nonlinear system, affine in control, given by

\[ \dot{x} = f(x) + g(x)u, \]  

(1)

with \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( m < n \), and \( f, g \) smooth. The IDA-PBC methodology relies on matching the system (1) with a generalized Hamiltonian structure\(^1\)

\[ \dot{x} = F(x)\nabla \mathcal{H}(x), \]  

(2)

for some smooth scalar function\(^2\) \( \mathcal{H}(x) \) and a full-rank matrix \( F(x) \in \mathbb{R}^{n \times n} \) satisfying \( F(x) + F(x)^\top \leq 0 \).

\(^1\)Throughout the paper all vectors are column vectors, even the gradient \( \nabla_x (\cdot) := \partial \mathcal{H}(\cdot) / \partial x \). When clear from the context the subindex of the operator \( \nabla \) and the arguments of the functions are omitted. For a mapping \( h : \mathbb{R}^n \to \mathbb{R}^n \), we define the Jacobian matrix as \( \nabla h(x) := (\nabla h_1(x), \ldots, \nabla h_n(x))^\top \).

\(^2\)If \( x \in \mathbb{R}^n \) are the energy variables then \( \mathcal{H}(x) \) represents the stored energy.

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Assume there exist matrices $g^\dagger(x)$, $F(x)$ and a scalar function $\mathcal{H}(x)$ that verify the PDEs
\[
g(x)^\dagger f(x) = g(x)^\dagger F(x)\nabla \mathcal{H}(x),
\]
resulting from the matching of systems (2) and (1), where $g(x)^\dagger$ is a full-rank left annihilator of $g(x)$, i.e. $g(x)^\dagger g(x) = 0$, and $\mathcal{H}(x)$ is such that
\[
x_* = \arg\min \mathcal{H}(x),
\]
with $x_* \in \mathbb{R}^n$ an equilibrium to be stabilized. We assume without loss of generality that $x_* = 0$. Then, the open-loop system (1) with the feedback
\[
u(x) = (g(x)^\dagger g(x))^{-1} g(x)^\dagger (-f(x) + F(x)\nabla \mathcal{H}(x)),
\]
takes the form (2) and is asymptotically stable if, in addition it is proved that $x_*$ is attractive, possibly invoking LaSalle’s Invariance Principle.

The key step in the design procedure relies on solving the PDEs given by (3) which is, usually, a difficult task (see [11] and [12] for some examples).

### III. Main Result

In this section an approximating integral and a dynamic extension are designed as an alternative to solve the PDEs in (3). The integral design is based on solving algebraically (3) for $\nabla \mathcal{H}(x)$. For, define the mapping $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $h(x) := [h_1(x), h_2(x), ..., h_n(x)]^\top$, which solves the algebraic equations (AEs) given by
\[
g(x)^\dagger f(x) = g(x)^\dagger F(x)h(x),
\]
and define the map $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ as
\[
H(x, \hat{x}) := \sum_{i=1}^{n} \int_{0}^{x_i} h_i(\hat{x}) \big|_{\hat{x}=x_i} \, ds,
\]
and the vector of errors $e := \hat{x} - x$. In what follows we describe the system in $(x, e)$-coordinates. Note that, the partial derivative of (7) with respect to $x$ is given by
\[
\nabla_x H(x, e) = h(x) + \nabla_x H(x, e) - h(x),
\]
where $h(x) = \nabla_x H(x, e)|_{e=0}$ and the matrix $\delta(x, e)$ is defined via (8). Consider now the closed-loop dynamics generated by (7), replacing in (5) the gradient $\nabla \mathcal{H}(x)$ with $\nabla_x H(x, e)$, namely
\[
\dot{x} = f(x) + g(x)u = f(x) + G(x)(-f(x) + F(x)\nabla_x H(x, e)) = f(x) + G(x)(-f(x) + F(x)h(x)) + G(x)F(x)\delta(x, e) = F(x)h(x) + S(x, e),
\]
where $G(x) := g(x)(g(x)^\dagger g(x))^{-1} g(x)^\dagger$ and $S(x, e) := G(x)F(x)\delta(x, e)$.

The following assumption is instrumental to derive the main result.

**Assumption 1**: The mapping $x \rightarrow h(x)$ is a diffeomorphism in a domain $D$ (possibly $D \equiv \mathbb{R}^n$) containing $x = 0$.

The Assumption 1 and the closed-loop dynamics (9) identify the equilibrium as $\hat{x}_{e=0} = F(x)h(x)$ and since $F(x)$ is a full-rank matrix then the equilibrium is at $h(x) = 0$.

**Proposition 1**: Suppose there exist symmetric and positive definite matrices $\Gamma$ and $K$ and, a full-rank matrix $F(x)$, such that
\[
\Gamma \nabla h(x)F(x) + (\Gamma \nabla h(x)F(x))^\top \leq 0, \quad \forall x \in D,
\]
with $h(x)$ a solution of (6) satisfying the Assumption 1. Then the equilibrium $(x, e) = (0, 0)$ of the system
\[
\dot{x} = \frac{F(x)}{e} - S(x, e) \Gamma h(x, e), \quad h(x, e) = 0,
\]
is (locally) stable in $(x, e) \in D \times \mathbb{R}^n$, with Lyapunov function
\[
V(x, e) := \frac{1}{2} h(x, e)^\top \Gamma h(x, e) + \frac{1}{2} |e|^2.
\]

**Proof.** Note that
\[
\dot{V} = h(x, e)^\top \Gamma \nabla h(x)\dot{x} + e^\top \dot{e} = h(x, e)^\top \Gamma \nabla h(x/F(x)h(x) + S(x, e)) + e^\top \dot{e} = \frac{1}{2} h(x, e)^\top (\Gamma \nabla h(x)F(x) + (\Gamma \nabla h(x)F(x))^\top) h(x) + e^\top (S(x, e)^\top \nabla h(x)^\top \Gamma h(x, e) + \dot{e}),
\]
and choosing the error dynamics as in (11) yields
\[
\dot{V} = \frac{1}{2} h(x, e)^\top (\Gamma \nabla hF(x) + (\Gamma \nabla hF(x))^\top) h(x) - e^\top Ke.
\]
Therefore (14) is negative (semi) definite if condition (10) is satisfied.

#### A. Linear systems

In this subsection we apply the main result of Proposition 1 to linear systems. This yields a Lyapunov-like interpretation of condition (10). Let $f(x) = Ax$ and $g(x) = b$, for some $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^{n \times m}$. In this case the general solution of (6) is given by $h(x) := Qx$ for some constant and full-rank matrices $Q$ and $F$. Thus (6) reads
\[
b^\top (A - FQ)x = 0, \quad \forall x \in D.
\]

The function (7) has a closed form given by
\[
H(x, \hat{x}) = \frac{1}{2} x^\top \text{diag}(Q)x + x^\top (Q - \text{diag}(Q))\hat{x},
\]
and the Lyapunov function (12) proposed for the nonlinear case becomes
\[
V(x, e) = \frac{1}{2} (Qx)^\top \Gamma (Qx) + \frac{1}{2} |e|^2.
\]
Therefore, the stability condition (10) reads
\[
\Gamma (QF) + (QF)^\top \Gamma \leq 0,
\]
which is a Lyapunov inequality and hence, if the matrix $(QF)$ is Hurwitz there is $\gamma > 0$ solving (15).

**Remark 1:** The previous result for linear systems includes the case of the standard IDA-PBC approach. In fact, selecting $Q = Q^\top > 0$ and defining $\Gamma := Q^{-1}$ then inequality (15) becomes $F + F^\top \leq 0$, which is the main assumption of the IDA-PBC approach. The case of linear systems, but in another context, was else thoroughly discussed in [10].

**B. A simple example**

Consider a nonlinear system of the form (1) with

$$ f(x) = \begin{bmatrix} x_2 + x_1^2 \\ x_1 + 2x_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x \in \mathbb{R}^2 \text{ and } u \in \mathbb{R}. $$

The algebraic equation (6) reads

$$ x_2 + x_1^2 = f_{11}h_1(x) + f_{12}h_2(x). $$(17)

Let

$$ F(x) = \begin{bmatrix} f_{11}(x) \\ f_{21}(x) \\ f_{12}(x) \\ f_{22}(x) \end{bmatrix}, $$

with $f_{11}$ and $f_{12}$ constants. Thus, the function (7) becomes

$$ H(x, \dot{x}) = \int_{0}^{x_1} h_1(\hat{x}_x)_{x_2 = s} \, ds 
+ \int_{0}^{x_2} \dot{x}_2 + \dot{x}_1^2 - f_{11}h_1(\hat{x})_{x_2 = s} \, ds. $$

Picking $h_1(\hat{x}) = \hat{x}_1$ and $f_{12} \neq 0$ yields

$$ H(x, \dot{x}) = \frac{x_1^2}{2} + \frac{x_2^2}{2f_{12}} + \frac{\dot{x}_2^2 - f_{11} \dot{x}_1}{2f_{12}} x_2, $$

which is not positive definite, although $H(x, \dot{x})\big|_{x=x}$ is locally positive definite. In this case it is enough to select $\Gamma = I_2$ and

$$ F(x) = \begin{bmatrix} -1 \\ 2x_1 + 1 \\ -\rho - x_1^2 - (x_1 + 1)^2 \end{bmatrix}, \quad (19) $$

for some $\rho > 1/4$. Note that $F + F^\top < 0$. The condition (10) for stability becomes

$$ \begin{bmatrix} -1 \\ \frac{1}{2} \\ -\rho - 2x_1^2 \end{bmatrix} \leq 0, \quad \text{for all } x \in \mathbb{R}^2, $$

hence the equilibrium $(x, e) = (0, 0)$ is globally asymptotically stable. In Fig. 1 and Fig. 2 a batch of 20 simulations for random initial conditions $(x(0), e(0))$ are shown. Fig. 1 and Fig. 2 show the time histories of the norm of the errors (top) and of the Lyapunov function (bottom).

**Remark 2:** The solution of the PDE (3) for $f_{11}$ and $f_{12}$ constants is

$$ \mathcal{H}(x) = \frac{x_1^2}{3f_{11}} + \frac{x_2^2}{2f_{12}} + \Psi \left( x_2 - \frac{f_{12}}{f_{11}} x_1 \right). $$

Therefore, we cannot construct a radially unbounded Lyapunov function selecting a particular $\mathcal{H}$, hence the IDA-PBC methodology can only be used for a local design.

3It is interesting to underscore that $H(x, \dot{x})\big|_{x=x}$ is not a solution of the PDE (3).

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**IV. LOWER-TRIANGULAR SYSTEMS**

The procedure of Section III.B can be used to design control laws for a class of strict-feedback systems (see [5] and [9]). This class is usually controlled by backstepping technique [5]. Consider the class of systems given by

$$ \begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) \\
& \vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_1, \ldots, x_n) \\
\dot{x}_n &= a(x) + b(x)u,
\end{align*} $$

where $x := (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $f_i : x_{i+1} + f_i(x_1, \ldots, x_{i+1})$ with $f_i(0) = 0$ for $i = 1, \ldots, n-1$, $b(0) \neq 0$ and $u \in \mathbb{R}$.
Select the matrix $F(x)$ as
\[
F(x) := \begin{bmatrix}
  f_{11} & 1 & 0 & 0 & \cdots & 0 \\
  f_{21} & f_{22} & 1 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  f_{n1} & \cdots & \cdots & \cdots & \cdots & f_{nn}
\end{bmatrix},
\]
and the vector $h(x)$ as
\[
h_i = f_{i-1} - \sum_{j=1}^{i-1} f_{(i-j)j} h_j, \quad i = 2, \ldots, n
\]
with $h_1(\hat{x}) := \hat{x}_1$. This choice provides a lower-triangular matrix $\nabla h F$ (with $\Gamma = I_n$) and allows to cancel all the off-diagonal terms with the $n(n-1)/2$ free off-diagonal parameters of the matrix $F$ solving the $n(n-1)/2$ equations
\[
(\nabla h F)_{i(i+1)} = -1, \quad i = 1, \ldots, n-1,
\]
\[
(\nabla h F)_{ij} = 0, \quad i = 3, \ldots, n; j = 1, \ldots, i-2.
\]
Thus, it only remains to select the diagonal terms $f_{ii}(x) < 0$, $i = 1, \ldots, n$, such that the matrix $\nabla h F$, now a diagonal matrix, is negative definite.

A. Example

Consider a 3-dimensional system with
\[
f(x) = \begin{bmatrix}
x_2 + x_1^2 \\
x_3 \\
x_2 + 2x_3
\end{bmatrix}, \quad g(x) = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}.
\]
Define the matrix $F(x)$ as
\[
F(x) := \begin{bmatrix}
f_{11} & f_{12} & f_{13} \\
f_{21}(x) & f_{22}(x) & f_{23}(x) \\
0 & f_{31}(x) & f_{32}(x) & f_{33}(x)
\end{bmatrix},
\]
with $f_{11}$, $f_{12}$ and $f_{23}$ constants. The AEs (6) read
\[
x_2 + x_1^2 = f_{11} h_1(x) + f_{12} h_2(x)
\]
\[
x_3 = f_{21} h_1(x) + f_{22} h_2(x) + f_{23} h_3(x).
\]
Thus, the function (7) becomes
\[
H(x, \hat{x}) = \int_{x_1 = s}^{x_1} h_1(\hat{x}) \, ds \\
+ \int_{x_2 = s}^{x_2} \frac{x_2 + x_1^2 - f_{11} h_1(x)}{f_{12}} \, ds \\
+ \int_{x_3 = s}^{x_3} \frac{x_2 + x_1^2 - f_{21} h_1(x) - f_{22} h_2(x)}{f_{23}} \, ds
\]
To show the flexibility of the approach we solve this example using the example in Section III.B as a partial solution and then using partially the system of equations (24)–(25) proposed in this section. Thus, pick $h_1(\hat{x}) = \hat{x}_1$ and let
\[
F(x) = \begin{bmatrix}
-1 & 0 & 0 & \cdots & 0 \\
2x_1 + 1 & -1 - x_1^2 - (x_1 + 1)^2 & 0 & 1 \\
0 & f_{32}(x) & f_{33}(x)
\end{bmatrix},
\]
for some $\rho > 1/4$. Notice that we still have $f_{31}$, $f_{32}$ and $f_{33}$ as free parameters to complete the design. Moreover, solving (24)–(25) for $f_{31}$ and $f_{32}$ yields
\[
\alpha(x) := 4x_1^2 + 4x_1 + \rho + 1 + 2x_2
\]
\[
f_{31} = -(4x_1 + 1) + (2x_1 + 1)(\alpha(x) + f_{22}(x_1))
\]
\[
f_{32} = -(4x_1 + 1) - (2x_1 + 1)\alpha(x) + f_{22}(x_1) - 1,
\]
and setting $\Gamma = I_3$ the condition (10) becomes
\[
\begin{bmatrix}
-1 & \frac{1}{x_1} & 0 \\
\frac{1}{\rho} - 2x_1^2 & 0 & 0 \\
0 & -f_{22}(x_1) + f_{33}(x)
\end{bmatrix} < 0.
\]
The condition (31), together with $F + F^T \leq 0$ from (30), allows to select the parameter $f_{33}$ as
\[
f_{33} := -\frac{\beta(x)}{\rho + x_1^2} + f_{22} < 0
\]
with
\[
\beta(x) := \begin{bmatrix}
f_{31} \\
1 + f_{32}
\end{bmatrix}^T \begin{bmatrix}
-f_{22} & x_1 + 1 & 1 \\
1 + f_{32}
\end{bmatrix} > 0.
\]
Therefore, since (31) holds for all $x \in \mathbb{R}^3$ the equilibrium $(x, \epsilon) = (0, 0)$ is globally asymptotically stable.

Remark 3: We underscore an important difference with respect to the original IDA-PBC approach. In IDA-PBC the fact that the full-rank matrix $F(x)$ is such that $F + F^T \leq 0$ is instrumental. In the example, the selection (32) implies the condition $F + F^T \leq 0$, which is not necessary, since the only condition for stability is (10). Hence, in this example the free parameter $f_{33}$ can be chosen to be negative and such that $f_{33}(x) < f_{22}(x_1)$ (instead of (32)) to satisfy only (31). For example we could select
\[
f_{33} := \gamma f_{22} < 0,
\]
for some $\gamma > 1$. Obviously, the design without the requirement $F + F^T \leq 0$ is simpler than the previous one (just compare (33) with (32)).

Remark 4: In this example the original set of PDEs (3) for any $f_{11}$ and $f_{12}$ (not necessarily constants) and $f_{13}$ constant is inconsistent, i.e. there is no function $H(x)$ that satisfies the PDEs.

We display the result of a batch of simulations for random initial conditions in $(x(0), e(0))$. Figures 3, 4, 5 and 6 show the time histories of the norm of the errors (top) and the Lyapunov function (bottom). In Fig. 3 and Fig. 4 the function $f_{33}$ is given by (32), and in Fig. 5 and Fig. 6 the function $f_{33}$ is given by (33).

V. UPPER-TRIANGULAR SYSTEMS

The approach proposed allows to solve control problems for classes of strict-feedforward systems [9]. To illustrate this issue consider a system with
\[
f(x) = \begin{bmatrix}
x_2 + x_3^2 \\
x_3 \\
0
\end{bmatrix}, \quad g(x) = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
\]
In this example we propose directly an explicit solution for the equations (35)–(36). For, define \( F(x) \) as
\[
F(x) := \begin{bmatrix} -1 & 1 & x_3^2 \\ 0 & 0 & 1 \\ 0 & -\rho & -2 \end{bmatrix},
\]
which is full-rank for \( \rho \neq 0 \). Thus, with this selection a solution of (35)–(36) is \( h(x) = [x_1, x_1 + x_2, x_3] \). In this case, we first check that \( \nabla h F \) is Hurwitz for all \( x \) and then solve the Lyapunov inequality for \( \Gamma > 0 \). Note that
\[
\nabla h(x) F(x) = \begin{bmatrix} -1 & 1 & x_3^2 \\ -1 & 1 & 1 + x_3^2 \\ 0 & -\rho & -2 \end{bmatrix},
\]
with the characteristic polynomial
\[
p(\lambda) = \lambda^3 + 2\lambda^2 + \rho(1 + x_3^2) \lambda + \rho.
\]
Note that \( \frac{d}{\lambda} p(\lambda) > 0 \) for \( \rho > 4/3 \), i.e. \( p(\lambda) \) is strictly monotonic; and that \( p(0) = \rho \) and \( p(-2) = -\rho(1 + 2x_3^2) \). This means that \( p(\lambda) \) has a real root for \( \lambda \in [-2, 0] \) and this is the only real root\(^4\). A detailed and straightforward analysis shows that the bound \( \rho > 4/3 \) obtained for \( x_3 = 0 \) is enough.

\(^4\)This can be seen using the discriminant of \( p(\lambda) \).
to guarantee that the real part of the two imaginary roots are negative for all \( x_3 \in \mathbb{R} \) and then, the design is completed. Thus, since for \( x_3 = 0 \) the matrix \( \nabla hF \) is constant \( \Gamma \) is obtained from the solution of (15). Finally, we can conclude that the equilibrium \( (x, e) = (0, 0) \) is asymptotically stable.

Remark 5: In this example the original set of PDEs (3) is again inconsistent hence there is no function \( \mathcal{H}(x) \) that satisfies the PDEs (3).

VI. CONCLUSIONS

A methodology to replace nonlinear PDEs by algebraic inequalities in the context of IDA-PBC is proposed. To this end an approximating integral together with a dynamic extension are designed. Main differences with the standard IDA-PBC methodology are highlighted through examples. Some connections with linear control theory and nonlinear cascade control techniques are also provided.

VII. ACKNOWLEDGMENTS

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