Abstract—This paper presents a constructive methodology to design sliding mode surfaces for a class of mechanical underactuated systems. The surfaces are designed taking advantage of a fictitious output and the Lyapunov theory. The fictitious output is chosen in such a way that the associated zero dynamics are minimum-phase. The attained nonlinear surface is a combination of the fictitious output and a nonlinear external controller designed using the Lyapunov theory. The technique is developed for a class of mechanical systems with uncertainty in their physical parameters, mainly in the inertial terms. In this class of systems we can find as examples some pendula like the pendulum on a cart and the Inertia Wheel Pendulum, which are analyzed. Simulations show the good performance, namely, time response and parametric robustness, of the controller.

I. INTRODUCTION

This paper presents a control scheme in which three classical methods, feedback linearization, design of control Lyapunov functions and sliding mode control, are combined to achieve the robust stabilization of the equilibrium point of a class of underactuated mechanical systems.

On the one hand sliding mode control (SMC) is a powerful and robust control methodology that has been widely studied in the last forty years [1], [2], [3], [4]. The sliding mode control approach is an efficient tool to design robust controllers for systems operating under uncertainty conditions. Sliding mode control has several advantages respect to other control techniques. Its main advantage is the low sensitivity to error estimation in the parameters of a plant and to disturbances which eliminates the necessity of exact modeling. When plants include sufficient information about the uncertainty, such as an upper bound, a robust control is normally designed. In this research line the stabilisation problem has been studied in recent years for different classes of systems with uncertainties like the works [5], [6]. Other advantage of sliding mode control is that enables the decoupling of the overall system motion into independent partial components of lower dimension, which are easier to control. Besides, one interesting feature of sliding mode control is that control actions are discontinuous functions which can be used to stabilize some classes of nonlinear systems which are not stabilizable by continuous state feedback laws [7].

On the other hand, most control design approaches using sliding surfaces are based upon Lyapunov and Jacobian linearization methods. In the Lyapunov approach, it is very difficult to find a Lyapunov function to design a control law and stabilize the system. In its turn, the Jacobian linearization approach yields only local stability. For this reason, when designing a control for a system it is better to use combined techniques, using sliding mode control in conjunction with other methods like backstepping, flatness and other traditional control design methods. In this case, a controller based on sliding mode control is designed so that the state trajectories tends to an specified surface. Furthermore, these methods do not use any additional assumption for the existence of the sliding mode surface that guarantees the stability of the system. Thus, the combination of the techniques takes the best of both approaches.

In this paper we use an approach next to the combination of sliding mode control and flatness, in the sense that we use an artificial output to stabilize the system as done in [8], [9], [10], [11], [12], but with one important difference. We address the problem to design a constructive sliding surface by combining feedback linearization and Lyapunov design. The main difference between this approach and the preceding one, is that the method based on flatness is not valid if the relative degree of the system with respect to the artificial output is not the same than the order of the system. That is, we cannot use the first approach if there exist zero dynamics associated to the output, but we still can use the second approach that we present.

The control scheme proposed in this paper is the following: firstly we design a fictitious output which guarantees that the zero dynamics are asymptotically stable; secondly we design a control lyapunov function to design the external controller for the linearized system; and finally to cope with parametric uncertainties a nonlinear sliding surface is introduced.

Respect to the stability issues, the present approach assures, on the one hand, exponential stability of the equilibrium point, which implies rejection to external disturbances, and on the other hand, robust stability to parametric uncertainties.

This paper is organized as follows: Next section is devoted to review some background concepts concerning sliding mode control. Section III shows how to design the fictitious output and the external controller. In section IV the main contribution of the paper is presented, that is, the design of new sliding mode surfaces. Section V shows two applications where the new sliding surfaces are applied, and finally in the last section the major conclusions are summarized.

II. BACKGROUND: SLIDING MODE CONTROL

Consider the SISO system

\[ x^{(n)} = f(x) + g(x)u \]  

(1)
where \( u \) is the control input and \( x = [x, \dot{x}, \ldots, x^{(n-1)}]^{\top} \) is the state vector.

We assume that the function \( f(x) \) is not completely known, but its uncertainty is upper bounded by a continuous function of \( x \) that is known; in its turn, the control gain \( g(x) \) is of known sign and is bounded by a continuous function of \( x \) that is also known. The control problem is to make the state \( x \) track a specific time varying state \( x_d = [x_d, \dot{x}_d, \ldots, x_d^{(n-1)}]^{\top} \) in the presence of model imprecision on \( f(x) \) and \( g(x) \). For example, usually, the inertia of a mechanical system is only known with limited accuracy.

Let \( \tilde{x} = x - x_d \) be the error of the output \( x \), and let \( \tilde{x} = x - x_d = [\tilde{x}, \dot{\tilde{x}}, \ldots, \tilde{x}^{(n-1)}]^{\top} \) be the error vector. Besides, let us define a surface in the state-space \( \mathbb{R}^n \) by

\[
s(x, t) = \left( \frac{d}{dt} + \lambda \right)^{n-1} \tilde{x} = 0 \tag{2}
\]

where, if \( n = 2 \), \( s(x, t) = \dot{\tilde{x}} + \lambda \tilde{x} = 0 \).

The simplified problem of keeping the scalar \( s \) at zero can be achieved by choosing the control law \( u \) of the system \( (1) \) such that outside of the surface the expression

\[
\frac{1}{2} \frac{d}{dt}s^2 \leq -\kappa|s| \tag{3}
\]

is verified, where \( \kappa > 0 \). Equation \( (3) \) constrains trajectories to evolve towards the surface. Once on the surface, the system trajectories remain on it. That is to say, if the sliding condition \( (3) \) is satisfied, the surface becomes an invariant set. The surface verifying \( (3) \) is called sliding surface, and the behavior of the system once on it is known as the sliding mode.

The behavior of the system when the sliding condition \( (3) \) is satisfied is shown in Fig. 1 for \( n = 2 \). In this case, the sliding surface is a line in the phase portrait, which contains the point \( x_d = [x_d, \dot{x}_d]^{\top} \). Starting from any initial condition, the state trajectory reaches the surface in a finite time and then moves through the surface towards \( x_d \) exponentially.

![Fig. 1. Graphical interpretation for n=2.](image)

To sum up, the idea behind equations \( (2) \) and \( (3) \) is to choose a function, \( s(x, t) \), whose behavior is known, and then design the feedback control law \( u \) for \( (1) \), such that \( V = \frac{1}{2}s^2 \) behaves as a Lyapunov function of the closed-loop system, even in the presence of disturbances and parametric uncertainties.

### III. PRELIMINARIES: CONSTRUCTIVE FEEDBACK LINEARIZATION

In this paper we will address the control problem of a class of underactuated mechanical systems with two degrees of freedom \( (n = 2) \), and only one control input \( (m = 1) \). The Lagrange’s equations read

\[
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla U(q) + D(\dot{q}) = [0 \ 1]^{\top} \tau \tag{4}
\]

where \( M \in \mathbb{R}^{2 \times 2} \), is the symmetric and positive definite inertia matrix, \( U \in \mathbb{R} \) is the potential function, the matrix \( C \in \mathbb{R}^{2 \times 2} \) contains the Coriolis and centrifugal forces, and \( D \in \mathbb{R}^{2 \times 2} \) represents the viscous friction forces, and \( \tau \in \mathbb{R} \), the number of independent control inputs. We now proceed to define the class of mechanical systems for which we can explicitly solve the control problem. In fact, the class considered refers to systems with underactuation degree one, \( m = 1 \). In the following, we split the set of generalized coordinates into \( q = (z, x) \in \mathbb{R} \times \mathbb{R} \), where the \( z \)-coordinate represents the unactuated degree of freedom and the \( x \)-coordinate the actuated one. After this partition the Lagrange’s equations of motion can be written as

\[
\begin{bmatrix}
m_{11} & m_{12} \\
m_{12} & m_{22}
\end{bmatrix} \dot{\tilde{q}} + \begin{bmatrix} f_1(q, \dot{q}) \\ f_2(q, \dot{q}) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau, \tag{5}
\]

where now \( m_{11}, m_{12} \in \mathbb{R}, m_{22} \in \mathbb{R} \) and we have introduced the scalar functions \( f_1(q, \dot{q}) \in \mathbb{R} \) and \( f_2(q, \dot{q}) \in \mathbb{R} \), respectively. The first step of the approach presented here is to linearize partially the equations of motion \( (4) \), as done in \([13]\) where it was called collocated partial feedback. Indeed, following some calculations given in \([13] \) it is easy to see that the partially feedback-linearized system takes the affine in control form:

\[
\dot{\tilde{q}} = \begin{bmatrix}
-m_{11}(q)^{-1}f_1(q, \dot{q}) \\
m_{22} - m_{12}^{-1}q
\end{bmatrix} u, \tag{6}
\]

where the scalar function \( f_1(q, \dot{q}) \) was defined as

\[
f_1(q, \dot{q}) \triangleq e_1^{\top} [C(q, \dot{q})\dot{q} + \nabla U(q) + D(\dot{q})], \tag{7}
\]

and, as usual, \( e_i \) denotes the \( i \)-th vector of the \( n \)-dimensional Euclidean basis. The inner loop reads

\[
\tau = \begin{bmatrix}
-m_{12} \\
m_{11}
\end{bmatrix} \begin{bmatrix} f_1(q, \dot{q}) \\ f_2(q, \dot{q}) \end{bmatrix} + \begin{bmatrix} m_{22} - m_{12}^{-1} \end{bmatrix} u. \tag{8}
\]

It will be assumed that the elements of the inertia matrix \( m_{11} \) and \( m_{12} \) do not depend on the actuated coordinates, \( m_{22} \) satisfies that either it is a constant matrix or a function of the actuated coordinates, and that the potential function is of the form
\[ U(q) \triangleq V(z) + \Phi(X), \]

On the one hand, the “constructive” output is defined as
\[ \tilde{y}_0(z, \dot{q}) \triangleq [\xi(z)]^T \dot{q}, \quad \tilde{y}_0(\cdot) \in \mathbb{R}. \]  
and its derivative with respect to time is given by
\[ \dot{\tilde{y}}_0 = \dot{\xi}(z) \dot{z} + [\xi(z)]^T \ddot{q} = \frac{d^2\xi}{dz^2} \dot{z}^2 - \frac{F_1 \xi}{m_{11}} \left( \frac{\xi_{m_{12}}}{m_{11}} - 1 \right) u = \nu \]  
with the scalar function \( \xi(z) \) defined as
\[ \xi(z) \triangleq k_1 m_{12}(z), \]  
being \( k_1 > 0 \) a scalar constant. On the other hand, the output is redesigned in such a way that the zero dynamics are locally exponentially stable (LES) giving rise to
\[ \ddot{\eta}(z, \dot{q}) \triangleq \tilde{y}_0 + k_2 \int_0^z m_{12}(\mu) d\mu, \quad \eta(\cdot) \in \mathbb{R}. \]  
with \( k_2 > 0 \) a scalar constant, and its derivative
\[ \dot{\eta}(z, \dot{q}) = \dot{\tilde{y}}_0 + k_2 m_{12}(z) \dot{z}. \]  
Therefore, the redesigned controller from (11) given by
\[ u = u_0 + \Delta^{-1} k_2 m_{12} \dot{z} = \Delta^{-1} \left( k_1 \left( \frac{dm_{12}}{dz} \dot{z}^2 - \frac{m_{12}}{m_{11}} f_1 \right) + k_2 m_{12} \dot{z} - \nu \right) \]  
using as the external controller
\[ \nu = -k_3 \left( m_{12} \dot{z} + p\ddot{q} \right), \]  
with \( k_3 > 0 \) and \( p > 0 \) scalar constants, makes the origin of the system Lyapunov asymptotically stable and locally exponentially stable (LES) (see Proposition 1 in [14] for the proof).

**IV. MAIN RESULT**

The main contribution of this paper is the constructive design of a sliding surface for a class of underactuated mechanical systems to which the constructive feedback linearization has been previously applied. The sliding surface will allow to cope with the uncertainty in the cancelation of terms done in the process of linearization by feedback.

The design method can be summarized in the following proposition, where for the sake of simplicity we will only consider the case of two degrees of freedom (\( n = 2 \)), with only one control input (\( m = 1 \)).

**Proposition 1:** Given the system (14) expressed as
\[ \ddot{\eta} = \Delta(s) \dot{\eta} - \Delta(z) u, \quad \text{with} \quad \ddot{\eta} \in \mathbb{R} \]

it will be assumed that the function \( \Delta(s) \dot{\eta} - \Delta(z) u \) is not exactly known, being estimated by the function \( \alpha(s, \dot{\eta}) \), verifying that the error of estimation is bounded by
\[ A(z, \dot{q}) \geq |\alpha(s, \dot{q}) - \alpha(z, \dot{q})| \]

Then, the sliding surface defined as
\[ s = \dot{\eta} - \int_0^t \nu d\tau, \quad \text{with} \quad s \in \mathbb{R} \]  
with a modified linearizing law given by
\[ u = u_L + \Delta^{-1} k_4 \text{sgn}(s), \quad \text{where} \quad k_4 \in \mathbb{R} \]

satisfies
\[ k_4(z, \dot{q}) > A(z, \dot{q}) \geq |\alpha(z, \dot{q}) - \alpha(z, \dot{q})| > 0, \]  
and \( u_L \) is the original constructive linearizing law, stabilizes asymptotically the origin of the system in the presence of parametric uncertainties in the model used in the feedback linearization process.

**Proof:** We want to proof that the state \( s \) tends asymptotically to zero. For that, we choose as a Lyapunov function candidate \( V = \frac{1}{2} s^2 \). Computing the derivative of \( V \) respect to time yields
\[ \dot{V} = s \dot{s} = s(\ddot{\eta} - \nu). \]  
Given the system \( \ddot{\eta} = \alpha(z, \dot{q}) - \Delta(z) u \), and considering a parametric uncertainty in \( \alpha(z, \dot{q}) \) bounded by \( |\alpha - \alpha| \leq A \), we can rewrite
\[ \dot{V} = s \dot{s} = s(\ddot{\eta} - \Delta(z) u - \nu). \]  
At this point we redefine the linearizing law, using an extra term in order to dominate the effect of the parametric uncertainty. In this way the new linearizing law takes the form \( u = u_L + \Delta^{-1} K_4 \text{sgn}(s) \) and the derivative of \( V \) yields
\[ \dot{V} = s \dot{s} = s(\ddot{\eta} - \alpha) - K_4 \text{sgn}(s) \leq \text{As} - K_4 |s| \leq (A - K_4) |s| \]  
Taking \( K_4 > A \) we ensure that the state \( s \) tends to zero asymptotically. The proof ends computing the explicit value of \( s \), that defines the control term \( K_4 \text{sgn}(s) \). Integrating respect to time yields
\[ s = \dot{\eta} + K_3 P \dot{\eta} + K_3 \int_0^z m_{12}(\mu) d\mu, \]  
where the integral term is well-defined if the assumptions in section III are satisfied. 

\[ \blacksquare \]
Remark 1: The result in proposition 1 holds if any sigmoid function of \( s \) is used to obtain a smooth approximation of \( \text{sign}(s) \), for instance \( \tanh(s) \). This feature will be used in section V to eliminate the chattering phenomenon.

Remark 2: The result in proposition 1 is only valid if the uncertainty is located in the term \( \bar{\alpha}(z, \dot{q}) \). For the general case, where both terms \( \bar{\alpha}(z, \dot{q}) \) and \( \Delta(z) \), are not exactly known, we present the following proposition.

Proposition 2: Given the system (14) expressed as
\[
\ddot{\eta} = \bar{\alpha}(z, \dot{q}) - \Delta(z)u, \quad \text{with} \quad \dot{\eta} \in \mathbb{R}
\]
it will be assumed that the functions \( \bar{\alpha}(z, \dot{q}) \) and \( \Delta(z) \) are not exactly known, being estimated by the functions \( \alpha(z, \dot{q}) \) and \( \Delta(z) \) respectively, verifying that the error of estimation in both functions is bounded and the sign of the control gain is known.

Then, the sliding surface defined as
\[
s = \dot{\eta} - \int_0^t \nu \, dr, \quad \text{with} \quad s \in \mathbb{R}
\]
where
\[
\dot{\eta}(z, \dot{q}) = [k_1 m_{12}(z)] \dot{q} + k_2 \int_0^z m_{12}(\mu) \, d\mu,
\]
\( \dot{\eta}(\cdot) \in \mathbb{R} \)
and
\[
\nu = -k_3 (m_{12}(z) \dot{z} + pq)
\]
with a modified linearizing law given by
\[
u = u_L + \Delta^{-1} k_4 (\bar{\alpha}(z, \dot{q}) \text{sign}(s), \quad \text{where} \quad k_4 \in \mathbb{R}
\]
\[
k_4(z, \dot{q}) \geq B |(\bar{\alpha} - \alpha) + (\Delta - \Delta) u_L| > 0
\]
and
\[
B > \Delta \Delta^{-1} > 0
\]
being \( u_L \), the original constructive linearizing law, stabilizes asymptotically the origin of the system in the presence of parametric uncertainties in the model used in the feedback linearization process.

Proof: Proceeding in a similar way than in the previous proof, we start from a Lyapunov function candidate
\[
V = \frac{1}{2} s^2.
\]
Computing the derivative of \( V \) respect to time yields
\[
\dot{V} = s \dot{s},
\]
where \( \dot{s} = (\dot{\eta} - \nu) \). Given the system \( \ddot{\eta} = \bar{\alpha}(z, \dot{q}) - \Delta(z)u \), and considering bounded parametric uncertainties in \( \alpha(z, \dot{q}) \) and \( \Delta(z) \), we can rewrite
\[
\dot{s} = \bar{\alpha}(z, \dot{q}) - \Delta(z)u - \nu.
\]
where \( u = u_L + \Delta^{-1} k_4 \text{sign}(s) \) and \( u_L = \Delta^{-1}(\alpha - \nu) \).

Then, the \( \dot{s} \) yields
\[
\dot{s} = \bar{\alpha} - \Delta^{-1}(\alpha - \nu + k_4 \text{sign}(s)) - \nu = \bar{\alpha} - \Delta^{-1}\alpha - (1 - \Delta^{-1}) \nu - \Delta^{-1} K_4 \text{sign}(s)
\]
Adding and subtracting \( \alpha \), collecting \((1 - \Delta^{-1})\), and substituting for \( u_L \), yields
\[
\dot{s} = (\bar{\alpha} - \alpha) + (1 - \Delta^{-1}) (\alpha - \nu) - \Delta^{-1} k_4 \text{sign}(s)
\]
\[
= (\bar{\alpha} - \alpha) + (\Delta - \Delta) u_L - \Delta^{-1} k_4 \text{sign}(s)
\]
\[
= (\bar{\alpha} - \alpha) - (\Delta - \Delta) u_L - \Delta^{-1} k_4 \text{sign}(s)
\]
\[
= \bar{\alpha} - u_L - \Delta^{-1} k_4 \text{sign}(s)
\]
Now, substituting in the derivative of \( V \) yields
\[
\dot{V} = ss = s (\bar{\alpha} - \Delta u_L - \Delta^{-1} k_4 \text{sign}(s)) = \leq |(\bar{\alpha} - \Delta u_L) - \Delta^{-1} k_4| s < 0
\]
In this way, assuming that the control gain is of known sign, that is \( \Delta^{-1} > 0 \), it is sufficient to take
\[
k_4 = B |\bar{\alpha} - \Delta u_L| \quad \text{with} \quad B > \Delta \Delta^{-1} > 0,
\]
to ensure that the state \( s \) tends to zero asymptotically. \( \blacksquare \)

Remark 3: In the case that \( \bar{\alpha} - \alpha = 0 \), \( k_4 \) should then verify to be
\[
k_4 > B |\Delta - \Delta| \Delta |u_L| = \varepsilon_r \Delta |u_L|,
\]
where \( \varepsilon_r \) is the relative error in the parameter estimation.

V. APPLICATIONS

In the following section two examples are presented as applications of the developed theory: the pendulum on a cart, where we will apply proposition 1, and the inertia wheel pendulum, where proposition 2 will be used.

A. Pendulum on a cart (PoC)

The dynamic equations of the pendulum on a cart can be written (see [16]) as follows
\[
ml^2 \ddot{x} + ml \cos z \ddot{x} - mgl \sin z = 0
\]
\[
(M + m) \ddot{x} + ml \cos z \ddot{x} - ml \sin z^2 = v.
\]

Using the collocated partial state feedback from [13] they can be expressed in a more compact form taking advantage
Now, substituting for \( \ddot{\theta} \) the derivative of the output in table 1, and we will compute first step we will assume that \( \dot{\theta} = 0 \) and deriving respect to time proceeded following the result of proposition 1. In this way we dynamics are non-trivial but at least minimum-phase.

In order to construct the sliding mode controller we will proceed following the result of proposition 1. In this way we start from the proposed surface

\[
s = \dot{\eta} + \eta_3(\eta n + ml \sin z),
\]

and deriving respect to time

\[
\dot{s} = \dot{\eta} + \eta_3(\eta \dot{n} + ml \cos z \dot{z}).
\]

Now we have to design the control law in two steps. In the first step we will assume that \( s = \dot{s} = 0 \). Then we will take the derivative of the output in table 1, and we will compute another derivative respect to time

\[
\eta = \dot{x} + k_1 m l \cos z \dot{z} + k_2 m l \sin z,
\]

\[
\dot{\eta} = \dot{x} + k_1 m l (- \sin z \dot{z}^2 + \cos z \dot{z}) + k_2 m l \cos z \dot{z}.
\]

Now, substituting for \( \ddot{\theta} \) and \( \dot{x} \) from (33) and (34) collecting \( u \) yields

\[
\dot{\eta} = (1 - k_1 m \cos^2 z)u + k_1 m l \sin z(- \dot{z}^2 + a \cos z) + k_2 m l \cos z \dot{z}.
\]

Substituting (37) in (36) for \( s = 0 \) yields the linearizing law \( u_L \)

\[
0 = (1 - k_1 m \cos^2 z)u_L + k_1 m l \sin z(- \dot{z}^2 + a \cos z) + k_2 m l \cos z \dot{z} + k_3 \beta \eta_3 \eta + ml \sin z)
\]

\[
u = u_L = \frac{u}{(1 - k_1 m \cos^2 z)^{-1}} k_4 \text{sign}(s)
\]

or other sigmoid function, for instance

\[
u = u_L = \frac{u}{(1 - k_1 m \cos^2 z)^{-1}} k_4 \tanh(s).
\]

Now taking as a Lyapunov function candidate \( V = \frac{1}{2} s^2 \), we have to verify that \( \dot{V} < 0 \).

\[
\dot{V} = s \dot{s} = \frac{s(a(q, \dot{q}) - \Delta(z) u - \nu)}{a - \frac{a}{2} k_1 m l \sin 2z - k_4 \text{sign}(s)} \leq (A - k_4) |s| \leq 0
\]

This is achieved by choosing

\[
k_4 > A = \frac{\bar{a}}{2} ml k_1.
\]

Simulations

In order to check the performance of the constructive sliding mode controller an experiment has been developed and shown in fig.2. It consists of changing the estimated value of the inertia of the pendulum in the control law until the performance of the closed loop degrades and an unstable behavior appears in the origin. In this case this gives rise to a limit cycle. In the second experiment, still in fig.2, the control law has been changed, introducing the new term \( k_4 \text{sign}(s) \). For the simulations we have chosen as parameters \( m = l = 1 \), \( k_1 = 10 \) and \( \bar{a} = 10 \) and \( a = 9.5 \). In this way the value of the gain has to be \( k_4 > 2.5 \). It can be observed in the phase portrait (fifth figure) how the states \( (\theta, \dot{\theta}) \) evolves along a non-smooth trajectory towards the origin. Relating to the control input \( u \), it can be observed the chattering phenomenon between the seconds 20 and 40 of the simulation, and how the frequency of this chattering is reduced from that point.

If the non-smooth input signal gives rise to a chattering phenomenon and represents more a drawback than an advantage it can be released by approximating the function \( \text{sign}(s) \) by any smooth sigmoid function, for instance \( \tanh(s) \). Using this function the new results can be seen in fig.3, where there is no chattering.

B. Inertia Wheel Pendulum (IWP)

This system is thoroughly described in \[15\], where the friction is not taken into account. In this section the friction model is included yielding

\[
\ddot{\theta} = a \sin z - k_1 \dot{\theta} \dot{\theta} - b u
\]

\[
\dot{\theta} = u.
\]

with \( a = \frac{ml}{J_1 + J_2} \), \( b = \frac{k_2}{J_1 + J_2} \), where \( J_i \) are inertia moments.

Following the same procedure as in the preceding example, we can attain the linearizing law

\[
u = u_L = -\Delta^{-1} \left( \nu + k_1 J_2 \left( \frac{k_1}{J_1 + J_2} \left( -ml \sin z + k_1 \dot{\theta} \right) - k_2 J_2 \dot{\theta} \right) \right)
\]

\[
\Delta = \left( 1 - k_1 J_2 \left( \frac{J_2}{J_1 + J_2} \right) \right)
\]

\[
u = -k_3 (P \eta + J_2 \dot{\theta})
\]

(39)
The final law to be applied using the sliding surface is

\[ u = u_L + \Delta^{-1} K_4(z, \dot{q}) \text{sign}(s), \quad \text{where} \]

\[ K_4(z, \dot{q}) \geq B \left| \dot{\alpha} + \Delta u_L \right| \quad \text{and} \quad B > \frac{\Delta}{\Delta} \] (40)

where \( K_4(z, \dot{q}) \) has to be estimated.

For the sake of simplicity we will assume that only the parameter \( J_1 \) is not well estimated. In that case

\[ B > \frac{\bar{J}_1 + J_2}{J_1 + J_2} \]

\[ \dot{\alpha} = K_1 J_2 mgl \sin z \left( \frac{1}{J_1 + J_2} - \frac{1}{\bar{J}_1 + J_2} \right) \] (41)

**Simulations**

For the simulations the following values of the parameters are considered: \( \bar{J}_1 = 4, J_3 = 2, \bar{J}_2 = J_2 = 1, mgl = 1 \). Figure 4 illustrates how the sliding mode control is able to cope with bounded parametric uncertainties when they affect to both terms \( \alpha \) and \( \Delta \), even when the uncertainty is big.

**VI. CONCLUSIONS**

In this paper we have shown a constructive methodology to design sliding mode surfaces for a class of underactuated mechanical systems. The surfaces have been designed by means of a fictitious output, its derivative and an extra nonlinear term attained from a control Lyapunov function. This surface ensures not only robust performance in the sense of parametric robustness but also in perturbation rejection, because the origin of this class of systems is exponentially stable. The technique has been developed successfully for a class of mechanical systems with uncertainty in the inertial terms. Finally, simulations have illustrated the good performance of the controller applied to both examples.

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REFERENCES