3 The Not-Quite Non-Atomic Games: Values and Cores of Large Games

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Non-atomic games were first introduced in several articles by Aumann and Shapley, which eventually culminated in their definitive book [1974, here denoted A/S]. We note that this was not the first treatment of games with a continuum of players; other articles, notably by Shapley et al., had done so. It is in A/S, however, that a rigorous definition of such games, and their meaning, first appears.

Essentially, a perfect market requires that any single trader’s actions have negligible effect on the payoffs of other traders. For example, if one dairy farmer increases or decreases his production of milk, this will not (presumably) affect the other farmers, who will still be selling their production at the same price. Nor will it affect consumers, who will be able to buy their normal quantity of milk at an unchanged price. This can only happen because the single farmer’s capacity is negligible by comparison with the size of the market.

In a similar way, in a general election (in a large voting constituency) any single voter’s choice of candidate (or, for that matter, the voter’s possible decision not to vote) will presumably have a negligible effect on the outcome of the election. Again, this can only happen if there is a very large number of voters (and no voter has more than a small number of votes).

In reality, of course, non-atomic games (whether market games or voting games) do not exist. By a simple induction argument, it is clear that, if no single player has a non-negligible effect, then no finite set of players has a non-negligible effect.

We can therefore think of non-atomic games only as limiting cases of very large games. Properties of non-atomic games are to be thought of as properties that hold, in the limit, for large games.

Under the circumstances, it is important to know the exact way in which these properties are obtained as the number of players increases (without bound). In general, it is clear that mere numbers are not enough: the way in which new players are brought into the game is very important.

As an example, we can consider one of the most interesting results of the A/S book: non-atomic production games. In these games, each player is given an endowment (a vector in $m$-space) and a production function that transforms $m$-vectors into utility. Coalitions can act by re-allocating their endowments so
as to maximize total production. A/S proves that, under rather broad and very reasonable conditions, these games (1) have a non-empty core, consisting (2) of a single point, which (3) coincides with the value of the game. Moreover, (4) this core can be expressed in terms of a vector of equilibrium prices, which are obtained (5) by an integration technique (the so-called diagonal process).

Now, under what conditions will large games have these properties (even approximately)? For example, property (3) seems to say that, for large economic/production/market games, the Shapley value will be close to the core. There is, in fact, a theorem (the value equivalence theorem) that states so. However, things are not as simple as this: certain conditions must be put on the players, their endowments, or their technology. We can no more say that—for arbitrary games—the Shapley value converges to the core, than probability theory can say that an arbitrary sum of many random variables converges to a normal variable.

In general, we have to think of these properties as holding, if at all, in the limit for large games, i.e., as the number of players in the game increases without bound. The simplest is to think of an infinite universe $\Omega$ of players; finite games grow in size as new players, all taken from $\Omega$, are brought into the game. Conditions can be put on $\Omega$. (For example, all members of $\Omega$ have at least 100, and not more than 1000, units of some good in their endowment, or they all like one good more than another, etc.)

We can then talk about properties that hold in the limit either in a strong sense—they will hold so long as new players are all taken from $\Omega$—or in a weaker sense—they hold if new players are taken from $\Omega$ only in a specific way. As an example of this weaker sense, we have the idea of market replication in which, starting from an original set $N$ of $n$ players, replicas of these players are brought into the game so that we only consider games with $kn$ players, $k$ of whom are identical (in their endowment, preferences, etc.) to each of the players in the original set $N$. (This is then known as the $k$-fold replicated game.)

As a very simple but popular example of this difference, we consider the shoe game (also known as the glove game in other authors’ articles). The universal set $\Omega$ contains infinitely many players, each given an endowment consisting of either one right shoe or one left shoe. Utility is created by forming pairs of shoes: each pair can be sold for $1. Thus the game has characteristic function

$$v(S) = \min \{|S \cap L|, |S \cap R|\},$$

where $L$ and $R$ are those subsets (of $\Omega$) consisting of left-shoe and right-shoe holders, respectively.

Suppose we wish to study property (3) above—coincidence of the core and the value. For finite games, this corresponds to saying that the value converges to the core, or (a slightly weaker statement) that the distance between the value and the core (measured, say, by the $L_\infty$ norm) converges to 0.
Now, it is easily proved that, starting from a specific set $N$ of players ($N \subset \Omega$), replication of the game will indeed give us the desired result, which we can put in the following terms:

Let $r$ and $l$ be, respectively, the number of left-hand and right-hand shoes in the original set $N$. Assume $r < l$; the core then consists of a single imputation $x$, given by $x_i = 1$ for each $i \in R$, and 0 for each $i \in L$.

Then, for any $\varepsilon > 0$, there exists an integer $h$ such that, for any $k > h$, the Shapley value $\varphi$ of the $k$-fold replicated game satisfies $\varphi_i > 1 - \varepsilon$ for all $i \in R$, and $\varphi_i < \varepsilon$ for each $i \in L$.

Thus, the convergence here is very much dependent on the way in which new players are brought into the game. If new players are introduced in a random manner, there is no guarantee of any type of convergence of the value.

Similar considerations will affect each of the five properties discussed above. In a series of papers dating to the early 1970’s, we have looked at these properties.

1. Is the core of large production games non-empty? Is there an $\varepsilon$-core with small positive $\varepsilon$? Non-emptiness of the core will not in general hold for finite production games unless the production functions are concave. In the non-concave case, an article by Flåm/Owen (2001) shows the existence of near-core imputations under reasonably broad conditions (broader, i.e., than replication of market); in the differentiable case, Saboýá/Owen (2002) gives a stronger result.

2. Will the core (at least generically) reduce to a single point? The core, when non-empty, does not always reduce to a single point. A article [Owen 1975] discusses conditions for both finite and infinite convergence of the core to a single point. This depends on non-degeneracy of the functions. Essentially, this article makes use of the replication technique.

3. The same article [Owen 1975] discusses conditions for the core to converge to imputations that correspond to equilibrium prices. Once again replication is necessary.

4. In [Owen 1972], the diagonal integration technique was developed as a technique for calculation of the Shapley value of finite games. For other types of games (e.g., weighted voting games), other authors have obtained similar results: in large games, a voter’s power (Shapley value) is proportional to his weight (number of votes). Once again, some strong regularity conditions are necessary for this to hold.