

## POWER IN WEIGHTED VOTING SYSTEMS

### TANNENBAUM ALGORITHM

```
<<DiscreteMath`Combinatorica`
```

```
<<Cooperat`Cooperat`
```

#### Banzhaf Power Index

The function `banzhafG` computes the generating function given a list of integer weights. The function `banzhafIndex` computes the total Banzhaf power index of player  $i$  by summing the appropriate coefficients in this generating function. Dividing each player's index by the sum of all the indices gives the Banzhaf power distribution.

P. Tannenbaum, Power in Weighted Voting Systems, The Mathematica Journal 7, 1 (1997) 58-63.

```
banzhafG[weights_List] := Times @@ (1 + x^weights)

Expand[banzhafG[{100,100,100,100,100,100,100}]]
Length[%]
1 + 7 x100 + 21 x200 + 35 x300 + 35 x400 + 21 x500 +
  7 x600 + x700
8

banzhafIndex[i_, weights_List, q_] :=
With[{g=banzhafG[Delete[weights,i]]},
  Sum[Coefficient[g,x,k],{k,q-weights[[i]],q-1}]]

banzhafPower[weights_List, q_] := (#1/Plus @@ #1 & ) [
  Table[banzhafIndex[i, weights, q], {i, Length[weights]}]]

banzhafIndexPlus[i_, weights_List, q_] :=
Module[{g, n, coefi}, g = banzhafG[Delete[weights, i]];
  coefi = CoefficientList[g, x]; n = Exponent[g, x] + 1;
  Plus @@ coefi[[Range[Max[1, q - weights[[i]] + 1],
    Min[q, n]]]]]

banzhafPowerPlus[weights_List, q_] :=
(#1/Plus @@ #1 & ) [Table[banzhafIndexPlus[i, weights, q],
  {i, Length[weights]}]]

votosUE = {10,10,10,10,8,5,5,5,5,4,4,3,3,3,2};
```

**Length4Expand4banzhafG4votosUE888**

86

**Timing [banzhafPower [votosUE, 62]]**

$$\{1.97 \text{ Second, } \left\{ \frac{1849}{16565}, \frac{1849}{16565}, \frac{1849}{16565}, \frac{1849}{16565}, \frac{1531}{16565}, \frac{973}{16565}, \right.$$

$$\left. \frac{973}{16565}, \frac{973}{16565}, \frac{973}{16565}, \frac{793}{16565}, \frac{793}{16565}, \frac{119}{3313}, \frac{119}{3313}, \frac{119}{3313}, \right.$$

$$\left. \frac{75}{3313} \right\}}$$
**Timing [banzhafPowerPlus [votosUE, 62]]**

$$\{0.66 \text{ Second, } \left\{ \frac{1849}{16565}, \frac{1849}{16565}, \frac{1849}{16565}, \frac{1849}{16565}, \frac{1531}{16565}, \frac{973}{16565}, \right.$$

$$\left. \frac{973}{16565}, \frac{973}{16565}, \frac{973}{16565}, \frac{793}{16565}, \frac{793}{16565}, \frac{119}{3313}, \frac{119}{3313}, \frac{119}{3313}, \right.$$

$$\left. \frac{75}{3313} \right\}}$$
**banUE62 = SetPrecision[%[[2]],3]**

$$\{0.112, 0.112, 0.112, 0.112, 0.0924, 0.0587, 0.0587,$$

$$0.0587, 0.0587, 0.0479, 0.0479, 0.0359, 0.0359, 0.0359,$$

$$0.0226\}$$
**Timing [banzhafPower [votosUE, 65]]**

$$\{1.92 \text{ Second, } \left\{ \frac{1227}{11149}, \frac{1227}{11149}, \frac{1227}{11149}, \frac{1227}{11149}, \frac{1033}{11149}, \frac{671}{11149}, \right.$$

$$\left. \frac{671}{11149}, \frac{671}{11149}, \frac{671}{11149}, \frac{507}{11149}, \frac{507}{11149}, \frac{411}{11149}, \frac{411}{11149}, \frac{411}{11149}, \right.$$

$$\left. \frac{277}{11149} \right\}}$$
**Timing [banzhafPowerPlus [votosUE, 65]]**

$$\{0.71 \text{ Second, } \left\{ \frac{1227}{11149}, \frac{1227}{11149}, \frac{1227}{11149}, \frac{1227}{11149}, \frac{1033}{11149}, \frac{671}{11149}, \right.$$

$$\left. \frac{671}{11149}, \frac{671}{11149}, \frac{671}{11149}, \frac{507}{11149}, \frac{507}{11149}, \frac{411}{11149}, \frac{411}{11149}, \frac{411}{11149}, \right.$$

$$\left. \frac{277}{11149} \right\}}$$

```
banUE65 =SetPrecision[%[[2]], 3]
{0.110, 0.110, 0.110, 0.110, 0.0927, 0.0602, 0.0602,
 0.0602, 0.0602, 0.0455, 0.0455, 0.0369, 0.0369, 0.0369,
 0.0248}
```

## Shapley-Shubik Power Index

The number of coalitions of weight  $k$  and size  $j$  is the coefficient of  $x^k z^j$  in the generating function  $g(x,y)$  for the Shapley-Shubik index. The function `ssG` gives the polynomial  $g(x,y)$ . The function `ssPower` computes the Shapley-Shubik power distribution.

```
ssG[weights_List] := Times @@ (1 + z*x^weights)

ssPower[weights_List, q_] :=
Module[{n = Length[weights], g, gg},
Table[g = ssG[Delete[weights, i]];
  gg = Sum[Coefficient[g, x, k], {k, q - weights[[i]],
    q - 1}]; Sum[Coefficient[gg, z^j]*j!*(n - j - 1)!,
  {j, 1, n - 1}], {i, n}]/n!]

ssPowerPlus[weights_List, q_] :=
Module[{n = Length[weights], g, gg, coefi},
Table[g = ssG[Delete[weights, i]];
  coefi = CoefficientList[g, x];
  gg = Plus @@ coefi[[Range[Max[1, q - weights[[i]] +
    1], q]]]; Sum[Coefficient[gg, z^j]*j!*(
  (n - j - 1)!, {j, 1, n - 1}], {i, n}]/n!]
```

```
Length[Expand[ssG[votosUE]]]
```

```
338
```

```
Timing[ssPower[votosUE, 62]]
```

```
{9.01 Second, { $\frac{7}{60}$ ,  $\frac{7}{60}$ ,  $\frac{7}{60}$ ,  $\frac{7}{60}$ ,  $\frac{860}{9009}$ ,  $\frac{19883}{360360}$ ,  $\frac{19883}{360360}$ ,
 $\frac{19883}{360360}$ ,  $\frac{19883}{360360}$ ,  $\frac{743}{16380}$ ,  $\frac{743}{16380}$ ,  $\frac{1588}{45045}$ ,  $\frac{1588}{45045}$ ,  $\frac{1588}{45045}$ ,  $\frac{932}{45045}$ }
}
```

```
Timing[ssPowerPlus[votosUE, 62]]
```

```
{2.03 Second, { $\frac{7}{60}$ ,  $\frac{7}{60}$ ,  $\frac{7}{60}$ ,  $\frac{7}{60}$ ,  $\frac{860}{9009}$ ,  $\frac{19883}{360360}$ ,  $\frac{19883}{360360}$ ,
 $\frac{19883}{360360}$ ,  $\frac{19883}{360360}$ ,  $\frac{743}{16380}$ ,  $\frac{743}{16380}$ ,  $\frac{1588}{45045}$ ,  $\frac{1588}{45045}$ ,  $\frac{1588}{45045}$ ,  $\frac{932}{45045}$ }
}
```

```
shaUE62 = SetPrecision[%[[2]], 3]
```

```
{0.117, 0.117, 0.117, 0.117, 0.0955, 0.0552, 0.0552,
 0.0552, 0.0552, 0.0454, 0.0454, 0.0353, 0.0353, 0.0353,
 0.0207}
```

```

Timing[ssPowerPlus[votosUE, 65]]

{2.04 Second, { $\frac{21733}{180180}$ ,  $\frac{21733}{180180}$ ,  $\frac{21733}{180180}$ ,  $\frac{21733}{180180}$ ,  $\frac{4216}{45045}$ ,
 $\frac{2039}{36036}$ ,  $\frac{2039}{36036}$ ,  $\frac{2039}{36036}$ ,  $\frac{2039}{36036}$ ,  $\frac{3587}{90090}$ ,  $\frac{3587}{90090}$ ,  $\frac{2987}{90090}$ ,  $\frac{2987}{90090}$ ,
 $\frac{2987}{90090}$ ,  $\frac{1667}{90090}$ }}

shaUE65 = SetPrecision[%[[2]], 3]

{0.121, 0.121, 0.121, 0.121, 0.0936, 0.0566, 0.0566,
0.0566, 0.0566, 0.0398, 0.0398, 0.0332, 0.0332, 0.0332,
0.0185}

USA = {"California", 54}, {"New York", 33}, {"Texas", 32},
{"Florida", 25}, {"Pennsylvania", 23},
{"Illinois", 22}, {"Ohio", 21}, {"Michigan", 18},
{"New Jersey", 15}, {"N. Carolina", 14},
{"Georgia", 13}, {"Virginia", 13},
{"Massachusetts", 12}, {"Indiana", 12},
{"Missouri", 11}, {"Wisconsin", 11}, {"Tennessee", 11},
{"Washington", 11}, {"Maryland", 10},
{"Minnesota", 10}, {"Louisiana", 9}, {"Alabama", 9},
{"Kentucky", 8}, {"Arizona", 8}, {"S. Carolina", 8},
{"Colorado", 8}, {"Connecticut", 8}, {"Oklahoma", 8},
{"Oregon", 7}, {"Iowa", 7}, {"Mississippi", 7},
{"Kansas", 6}, {"Arkansas", 6}, {"W. Virginia", 5},
{"Utah", 5}, {"Nebraska", 5}, {"New Mexico", 5},
{"Maine", 4}, {"Nevada", 4}, {"New Hampshire", 4},
{"Hawaii", 4}, {"Idaho", 4}, {"Rhode Island", 4},
{"Montana", 3}, {"S. Dakota", 3}, {"Delaware", 3},
{"N. Dakota", 3}, {"Vermont", 3}, {"Alaska", 3},
{"Wyoming", 3}, {"Dt. Columbia", 3}};

{states, USAVotes} = Transpose[USA];

p1 = SetPrecision[USAVotes/Plus @@ USAVotes, 3];

banUSA = SetPrecision[banzhafPowerPlus[USAVotes, 270], 3];

shaUSA = SetPrecision[ssPowerPlus[USAVotes, 270], 3];

TableForm[Transpose[{p1, banUSA, shaUSA}], TableHeadings ->
{states, {"Votes Index", "Banzhaf", "Shapley"}}]



|              | Votes Index | Banzhaf | Shapley |
|--------------|-------------|---------|---------|
| California   | 0.100       | 0.111   | 0.108   |
| New York     | 0.0613      | 0.0620  | 0.0630  |
| Texas        | 0.0595      | 0.0600  | 0.0609  |
| Florida      | 0.0465      | 0.0463  | 0.0469  |
| Pennsylvania | 0.0428      | 0.0425  | 0.0430  |
| Illinois     | 0.0409      | 0.0406  | 0.0411  |
| Ohio         | 0.0390      | 0.0387  | 0.0391  |


```

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Michigan	0.0335	0.0330	0.0333
New Jersey	0.0279	0.0275	0.0276
N. Carolina	0.0260	0.0256	0.0257
Georgia	0.0242	0.0238	0.0238
Virginia	0.0242	0.0238	0.0238
Massachusetts	0.0223	0.0219	0.0220
Indiana	0.0223	0.0219	0.0220
Missouri	0.0204	0.0201	0.0201
Wisconsin	0.0204	0.0201	0.0201
Tennessee	0.0204	0.0201	0.0201
Washington	0.0204	0.0201	0.0201
Maryland	0.0186	0.0182	0.0182
Minnesota	0.0186	0.0182	0.0182
Louisiana	0.0167	0.0164	0.0164
Alabama	0.0167	0.0164	0.0164
Kentucky	0.0149	0.0146	0.0145
Arizona	0.0149	0.0146	0.0145
S. Carolina	0.0149	0.0146	0.0145
Colorado	0.0149	0.0146	0.0145
Connecticut	0.0149	0.0146	0.0145
Oklahoma	0.0149	0.0146	0.0145
Oregon	0.0130	0.0128	0.0127
Iowa	0.0130	0.0128	0.0127
Mississippi	0.0130	0.0128	0.0127
Kansas	0.0112	0.0109	0.0109
Arkansas	0.0112	0.0109	0.0109
W. Virginia	0.00929	0.00910	0.00904
Utah	0.00929	0.00910	0.00904
Nebraska	0.00929	0.00910	0.00904
New Mexico	0.00929	0.00910	0.00904
Maine	0.00743	0.00728	0.00722
Nevada	0.00743	0.00728	0.00722
New Hampshire	0.00743	0.00728	0.00722
Hawaii	0.00743	0.00728	0.00722
Idaho	0.00743	0.00728	0.00722
Rhode Island	0.00743	0.00728	0.00722
Montana	0.00558	0.00546	0.00541

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S. Dakota	0.00558	0.00546	0.00541
Delaware	0.00558	0.00546	0.00541
N. Dakota	0.00558	0.00546	0.00541
Vermont	0.00558	0.00546	0.00541
Alaska	0.00558	0.00546	0.00541
Wyoming	0.00558	0.00546	0.00541
Dt. Columbia	0.00558	0.00546	0.00541