European Journal of Operational Research 196 (2009) 1008–1014

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Stochastics and Statistics

Axiomatizations of the Shapley value for games on augmenting systems

J.M. Bilbao *, M. Ordóñez

Department of Applied Mathematics II, University of Seville, Camino de los Descubrimientos, Escuela Superior de Ingenieros, 41092 Sevilla, Spain

ARTICLE INFO

Article history:
Received 31 October 2007
Accepted 24 April 2008
Available online 4 May 2008

Keywords:
Augmenting system
Shapley value

ABSTRACT

This paper deals with cooperative games in which only certain coalitions are allowed to form. There have been previous models developed to confront the problem of unallowable coalitions. Games restricted by a communication graph were introduced by Myerson and Owen. In their model, the feasible coalitions are those that induce connected subgraphs. Another type of model is introduced in Gilles, Owen and van den Brink. In their model, the possibilities of coalition formation are determined by the positions of the players in a so-called permission structure. Faigle proposed another model for cooperative games defined on lattice structures. We introduce a combinatorial structure called *augmenting system* which is a generalization of the antimatroid structure and the system of connected subgraphs of a graph. In this framework, the Shapley value of games on augmenting systems is introduced and two axiomatizations of this value are showed.

1. Introduction

The purpose of this paper is to develop a new framework in which to analyze cooperative games in which only certain coalitions are allowed to form. We will study the structure of such allowable coalitions using the theory of *augmenting systems*, a notion developed to combinatorial abstract theory. The first model in which the feasible coalitions are defined by the connected subgraphs of a graph is introduced by Myerson [14]. Contributions on graph-restricted games include Owen [15], Borm et al. [6] and Hamiache [12]. In these models the possibilities of coalition formation are determined by a *communication graph* between the players. Another type of combinatorial structure introduced by Gilles, Owen and van den Brink [11] and van den Brink [7] is equivalent to a subclass of antimatroids. This line of research focuses on the possibilities of coalition formation determined by the positions of the players in the *permission structures*. In addition, given a cooperative game and a set system of feasible coalitions, a *restricted game* is then defined by using the maximal feasible subsets of a coalition. The Shapley value [16] has been generalized by Myerson [14] for restricted games by communication situations, which are defined by a cooperative game and the family of all connected subgraphs of a graph. Bilbao [5] obtained explicit formulas for the Shapley value of games restricted by augmenting systems.

Let us consider a set system $(N,\mathscr{F})$, where $\mathscr{F}\subseteq 2^N$ is a family of feasible subsets of the player set N. An important fact is that all the above contributions are devoted to analyze standard cooperative games $v^{\mathscr{F}}:2^N\to\mathbb{R}$, named restricted games and defined by

$$v^{\mathscr{F}}(S)=\sum_{T\in C_{\mathscr{F}}(S)} v(T)\quad\text{for all}\quad S\subseteq N,$$

where $C_{\mathscr{F}}(S)$ is the set of maximal nonempty feasible subsets (components) of S and $C_{\mathscr{F}}(S)$ is a partition of a subset of S. Notice that $v^{\mathscr{F}}(S)=v(S)$ for all $S\in\mathscr{F}$. Furthermore, if $S\notin\mathscr{F}$ the definition of $v^{\mathscr{F}}$ assigns to S the sum of the outputs of feasible coalitions that players from S could jointly achieve.

In this paper we consider an augmenting system $(N,\mathscr{F})$ and a real-valued function $v:\mathscr{F}\to\mathbb{R}$ such that $v(\emptyset)=0$. Thus, our way of looking at the problem of unallowable coalitions is completely different and there is not overlap with the approach given by Bilbao [5]. Most closely related to our approach is the work of Faigle and Kern [10] on cooperative games under *precedence constraints*, which are games defined on a lattice of feasible subsets. Their model has been generalized by Bilbao and Edelman [3,4] to games on convex geometries. Since convex geometries and antimatroids are special case of augmenting systems our analysis follows the above mentioned papers quite closely. Note that in this approach, cooperative games are defined only in the set of feasible coalitions and the notion of restricted game makes no sense.

In Section 2, we recall the concept of augmenting system and describe its fundamental properties. Section 3 introduces games on augmenting systems and by using the classical approach of Weber [18], we obtain a characterization of the extended Shapley value under the axioms of linearity, dummy, efficiency and chain. Finally, in Section 4 we show a new axiomatization of the extended Shapley value of games on augmenting systems by using linearity,

* Corresponding author.
E-mail address: mbilbao@us.es (J.M. Bilbao).

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doi:10.1016/j.ejor.2008.04.028

dummy and efficiency axioms, and the hierarchical strength axiom instead the chain axiom.

2. Augmenting systems

Antimatroids were introduced by Dilworth [8] as particular examples of semimodular lattices. Since then, several authors have obtained the same concept by abstracting various combinatorial situations (see Korte et al. [13]). Let N be a finite set. A *set system* over N is a pair $(N, \mathscr{F})$ where $\mathscr{F} \subseteq 2^N$ is a family of subsets. The sets belonging to $\mathscr{F}$ are called *feasible*. We will write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$, respectively.

Definition 1. A set system $(N, \mathscr{A})$ is an antimatroid if

A1. $\emptyset \in \mathscr{A}$,
A2. for $S, T \in \mathscr{A}$ we have $S \cup T \in \mathscr{A}$,
A3. for $S \in \mathscr{A}$ with $S \neq \emptyset$, there exists $i \in S$ such that $S \setminus i \in \mathscr{A}$.

Let $(N, \mathscr{A})$ be an antimatroid and let $S, T \in \mathscr{A}$ such that $|S| < |T|$. Property A3 implies an ordering $T = \{i_1, \ldots, i_t\}$ with $\{i_1, \ldots, i_j\} \in \mathscr{A}$ for $j = 1, \ldots, t$. Let $k \in \{1, \ldots, t\}$ be the minimum index with $i_k \notin S$. Then $S \cup i_k = S \cup \{i_1, \ldots, i_k\} \in \mathscr{A}$ by property A2. Therefore, the definition of antimatroid implies the following *augmentation property*: If $S, T \in \mathscr{A}$ with $|S| < |T|$ then there exists $i \in T \setminus S$ such that $S \cup i \in \mathscr{A}$.

Convex geometries are a combinatorial abstraction of convex sets introduced by Edelman and Jamison [9].

Definition 2. A set system $(N, \mathscr{G})$ is a convex geometry if it satisfies the following properties:

G1. $\emptyset \in \mathscr{G}$,
G2. for $S, T \in \mathscr{G}$ we have $S \cap T \in \mathscr{G}$,
G3. for $S \in \mathscr{G}$ with $S \neq N$, there exists $i \in N \setminus S$ such that $S \cup i \in \mathscr{G}$.

We will introduce a new combinatorial structure as follows.

Definition 3. An augmenting system is a set system $(N, \mathscr{F})$ with the following properties:

P1. $\emptyset \in \mathscr{F}$,
P2. for $S, T \in \mathscr{F}$ with $S \cap T \neq \emptyset$, we have $S \cup T \in \mathscr{F}$,
P3. for $S, T \in \mathscr{F}$ with $S \subset T$, there exists $i \in T \setminus S$ such that $S \cup i \in \mathscr{F}$.

The relationship between the combinatorial structures above mentioned is given by Bilbao [5] in the next proposition.

Proposition 4

(i) *An augmenting system $(N, \mathscr{F})$ is an antimatroid if and only if $\mathscr{F}$ is closed under union.*
(ii) *An augmenting system $(N, \mathscr{F})$ is a convex geometry if and only if $\mathscr{F}$ is closed under intersection and $N \in \mathscr{F}$.*

Example. The following collections of subsets of $N = \{1, \ldots, n\}$, given by $\mathscr{F} = 2^N$, $\mathscr{F} = \{\emptyset, \{i\}\}$, where $i \in N$, and $\mathscr{F} = \{\emptyset, \{1\}, \ldots, \{n\}\}$, are augmenting systems over N.

Example. Let us consider a communication graph $G = (N, E)$, where N is the set of players and E is the set of edges which represents the bilateral communication between some players. Given a coalition $S \subseteq N$, the set of edges between players in S is denoted by $E(S) = \{ij \in E : i, j \in S\}$. Thus, the set system $(N, \mathscr{F})$ given by

$$\mathscr{F} = \{S \subseteq N : (S, E(S)) \quad \text{is a connected subgraph of} \quad G\}$$

is an augmenting system.

The next characterization of the augmenting systems derived from the connected subgraphs of a graph is proved by Algaba et al. [2]

Theorem 5. *An augmenting system $(N, \mathscr{F})$ is the system of connected subgraphs of the graph $G = (N, E)$, where $E = \{S \in \mathscr{F} : |S| = 2\}$ if and only if $\{i\} \in \mathscr{F}$ for all $i \in N$.*

Example. Gilles et al. [11] showed that the feasible coalition system $(N, \mathscr{F})$ derived from the conjunctive or disjunctive approach contains the empty set, the ground set N, and that it is closed under union. Algaba et al. [1] showed that the coalition systems derived from the conjunctive and disjunctive approach were identified to *poset antimatroids* and *antimatroids with the path property*, respectively. Thus, these coalition systems are augmenting systems.

Example. The set system given by $N = \{1, 2, 3, 4\}$ and

$$\mathscr{F} = \{\emptyset, \{1\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}$$
$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\}$$

is an augmenting system. Since $\{1, 4\} \notin \mathscr{F}$ the system $(N, \mathscr{F})$ is not an antimatroid. Moreover, $\{1, 2\} \cap \{2, 4\} = \{2\} \notin \mathscr{F}$ and hence $(N, \mathscr{F})$ is not a convex geometry.

Definition 6. Let $(N, \mathscr{F})$ be an augmenting system. For a feasible coalition $S \in \mathscr{F}$, we define the set $S^* = \{i \in N \setminus S : S \cup i \in \mathscr{F}\}$ of augmentations of S and the set $S^+ = S \cup S^* = \{i \in N : S \cup i \in \mathscr{F}\}$.

Proposition 7. *Let $(N, \mathscr{F})$ be an augmenting system. Then the interval $[S, S^+]_{\mathscr{F}} = \{C \in \mathscr{F} : S \subseteq C \subseteq S^+\}$ is a Boolean algebra for every nonempty $S \in \mathscr{F}$.*

Proof. It is suffices to show that $[S, S^+]_{\mathscr{F}} = \{C \subseteq N : S \subseteq C \subseteq S^+\}$, i.e. for every $C \subseteq N$ such that $S \subseteq C \subseteq S^+$ we have $C \in \mathscr{F}$. If $S^* = \emptyset$ then $[S, S^+]_{\mathscr{F}} = \{S\}$. Otherwise $S^* = \{i_1, \ldots, i_p\}$ and $S \subseteq C \subseteq S^+$ implies $C = S \cup \{i_1, \ldots, i_q\}$ for some $1 \leqslant q \leqslant p$. We prove that $C \in \mathscr{F}$ by induction on q. For $q = 1$ we know that $S \cup \{i_1\} \in \mathscr{F}$. Assume $S \cup \{i_1, \ldots, i_k\} \in \mathscr{F}$. Since $S \cup \{i_{k+1}\} \in \mathscr{F}$ and $(S \cup \{i_1, \ldots, i_k\}) \cap (S \cup \{i_{k+1}\}) = S \neq \emptyset$, property P2 yields $S \cup \{i_1, \ldots, i_k, i_{k+1}\} \in \mathscr{F}$. □

Definition 8. Let $(N, \mathscr{F})$ be an augmenting system. An element i in $S \in \mathscr{F}$ is an extreme point of S if $S \setminus i \in \mathscr{F}$.

The set of extreme points of S is denoted by $\text{ex}(S)$. Note that property P3 implies A3 and hence $|\text{ex}(S)| \geqslant 1$ for any nonempty $S \in \mathscr{F}$.

3. Axioms for the Shapley value

A cooperative game is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The players are the elements of N and the coalitions are the elements of the Boolean algebra 2^N.

Definition 9. A cooperative game on the augmenting system $(N, \mathscr{F})$ is a triple $(N, v, \mathscr{F})$, where $v : \mathscr{F} \to \mathbb{R}$ is a real-valued function such that $v(\emptyset) = 0$.

The coalitions are the feasible sets belonging to $\mathscr{F}$ and the players are the elements of N. Let $\Gamma(\mathscr{F})$ be the real vector space of the games on the augmenting system $\mathscr{F} \subseteq 2^N$. We will follow the work of Weber [18] to obtain an axiomatic development of the Shapley value for games on augmenting system. This way to extend the Shapley value is the logical path to obtain the adaptation of the classical axioms (linearity, dummy, efficiency, and symmetry) to cooperative games on combinatorial structures. For this, we consider the following game on $\mathscr{F}$. For any $T \in \mathscr{F}, T \neq \emptyset$, the *identity game* $\delta_T : \mathscr{F} \to \mathbb{R}$ is defined by

$$\delta_T(S) := \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{if } S \neq T. \end{cases}$$

Let $\Phi : \Gamma(\mathscr{F}) \to \mathbb{R}^n$ a map such that $\Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v))$. The meaning of this function is to give the expected payoffs to the players of a game. We introduce several axioms that give rise to a unique function for games on augmenting systems. If $\mathscr{F} = 2^N$ then this function is equal to the classical Shapley value. First, we consider the linearity property.

Linearity axiom: For all $\alpha, \beta \in \mathbb{R}$, and $v, w \in \Gamma(\mathscr{F})$ we have

$$\Phi_i(\alpha v + \beta w) = \alpha \Phi_i(v) + \beta \Phi_i(w) \quad \text{for every} \quad i \in N.$$

Theorem 10. *Let $\Phi_i : \Gamma(\mathscr{F}) \to \mathbb{R}$ be a value for i which satisfies the linearity axiom. Then there exists an unique set of coefficients $\{a_S^i : S \in \mathscr{F}, S \neq \emptyset\}$ such that*

$$\Phi_i(v) = \sum_{\{S \in \mathscr{F} : S \neq \emptyset\}} a_S^i v(S)$$

for every $v \in \Gamma(\mathscr{F})$.

Proof. The collection $\{\delta_S : S \in \mathscr{F}, S \neq \emptyset\}$ is a basis of the vector space $\Gamma(\mathscr{F})$. Then, for every game $v \in \Gamma(\mathscr{F})$

$$v = \sum_{\{S \in \mathscr{F} : S \neq \emptyset\}} v(S) \delta_S.$$

Let $a_S^i = \Phi_i(\delta_S)$ for every $i \in N$, and every nonempty $S \in \mathscr{F}$. Applying the linearity axiom we obtain

$$\Phi_i(v) = \sum_{\{S \in \mathscr{F} : S \neq \emptyset\}} a_S^i v(S),$$

for every $v \in \Gamma(\mathscr{F})$. $\square$

We will now introduce the concept of dummy player.

Definition 11. The player $i \in N$ is a dummy player in the game $v \in \Gamma(\mathscr{F})$ if for all $S \in \mathscr{F}$ such that $i \in S^*$, we have

$$v(S \cup i) - v(S) = \begin{cases} v(\{i\}) & \text{if } \{i\} \in \mathscr{F}, \\ 0 & \text{otherwise}. \end{cases}$$

This definition derives from the observation that a dummy player has no strategic role in the game, because of such a player contributes precisely $v(\{i\})$ or zero. We need a preparatory lemma about some properties of the dummy player in the identity game.

Lemma 12. *Let $(N, \mathscr{F})$ be an augmenting system and consider a nonempty $S \in \mathscr{F}$. Then:*

(i) *If $i \in S \setminus \text{ex}(S)$ then player i is dummy in the identity game δ_S.*
(ii) *If $i \notin S^+$ then player i is dummy in the identity game δ_S.*
(iii) *If $i \in S^*$ then player i is dummy in the game $\delta_S + \delta_{S \cup i}$.*

Proof

1. Note that if $\{i\} \in \mathscr{F}$ then $i \in \text{ex}(\{i\})$, and hence $S \neq \{i\}$. This implies $\delta_S(\{i\}) = 0$. Now let $C \in \mathscr{F}$ be such that $i \in C^*$, and it is sufficient to prove that $\delta_S(C \cup i) - \delta_S(C) = 0$. If $S = C \cup i$ then $C = S \setminus i \in \mathscr{F}$, so that $i \in \text{ex}(S)$, a contradiction. If $S = C$ then $i \in S^* = S^+ \setminus S$, which is a contradiction. Thus, $\delta_S(C \cup i) = 0$ and $\delta_S(C) = 0$.
2. If $S = \{i\} \in \mathscr{F}$ then $i \in S^+$, contradicting the hypothesis. Then $\delta_S(\{i\}) = 0$. Consider $C \in \mathscr{F}$ such that $i \in C^*$. Since $i \notin S^+ = S \cup S^*$ we have $S \neq C \cup i$ and $S \neq C$, because otherwise $i \in S$ or $i \in S^*$. Thus, $\delta_S(C \cup i) - \delta_S(C) = 0$.
3. If $\{i\} \in \mathscr{F}$ where $i \in S^*$ then $i \notin S$, and hence $S \neq \{i\}$. Since $S \neq \emptyset$ we also know that $S \cup i \neq \{i\}$. Then $\delta_S(\{i\}) + \delta_{S \cup i}(\{i\}) = 0$. We now take $C \in \mathscr{F}$ such that $i \in C^*$. Since $i \in S^* = S^+ \setminus S$ we obtain $S \neq C \cup i$ and $S \cup i \neq C$, because otherwise $i \in S$ or $i \in C$. Thus
$$(\delta_S + \delta_{S \cup i})(C \cup i) - (\delta_S + \delta_{S \cup i})(C) = \delta_{S \cup i}(C \cup i) - \delta_S(C).$$
Finally, the equivalence $S \cup i = C \cup i \iff S = C$ implies
$$\delta_{S \cup i}(C \cup i) - \delta_S(C) = 0$$
for all $C \in \mathscr{F}$ such that $i \in C^*$. $\square$

The following axiom gives the payoff received for a dummy player.

Dummy axiom: If the player $i \in N$ is a dummy in $v \in \Gamma(\mathscr{F})$, then

$$\Phi_i(v) = \begin{cases} v(\{i\}) & \text{if } \{i\} \in \mathscr{F}, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 13. *Let $\Phi_i : \Gamma(\mathscr{F}) \to \mathbb{R}$ be a value for player $i \in N$ that satisfies linearity and dummy axioms. Then, for every game $v \in \Gamma(\mathscr{F})$*

$$\Phi_i(v) = \sum_{\{T \in \mathscr{F} : i \in T^*\}} a_{T \cup i}^i [v(T \cup i) - v(T)].$$

Moreover, if $\{i\} \in \mathscr{F}$ then

$$\sum_{\{T \in \mathscr{F} : i \in T^*\}} a_{T \cup i}^i = 1.$$

Proof. We know from Theorem 10 that for a fix player $i \in N$

$$\begin{aligned} \Phi_i(v) &= \sum_{\{S \in \mathscr{F} : S \neq \emptyset\}} a_S^i v(S) \\ &= \sum_{\{S \in \mathscr{F} : i \in \text{ex}(S)\}} a_S^i v(S) + \sum_{\{S \in \mathscr{F} : i \notin S\}} a_S^i v(S) + \sum_{\{S \in \mathscr{F} : i \in S \setminus \text{ex}(S)\}} a_S^i v(S). \end{aligned}$$

Lemma 12 (i) implies that if $i \in S \setminus \text{ex}(S)$ then player i is dummy in the identity game δ_S. Applying dummy axiom we obtain $a_S^i = \Phi_i(\delta_S) = 0$, for all $i \in S \setminus \text{ex}(S)$. Moreover, $N \setminus S = S^* \cup (N \setminus S^+)$ and $S^* \cap (N \setminus S^+) = \emptyset$, and then we have

$$\begin{aligned} \Phi_i(v) &= \sum_{\{S \in \mathscr{F} : i \in \text{ex}(S)\}} a_S^i v(S) + \sum_{\{S \in \mathscr{F} : i \notin S\}} a_S^i v(S) \\ &= \sum_{\{S \in \mathscr{F} : i \in \text{ex}(S)\}} a_S^i v(S) + \sum_{\{S \in \mathscr{F} : i \in S^*\}} a_S^i v(S) + \sum_{\{S \in \mathscr{F} : i \notin S^+\}} a_S^i v(S). \end{aligned}$$

If $i \notin S^+$ then player i is dummy in the identity game δ_S by Lemma 12 (ii). Hence dummy axiom implies $a_S^i = \Phi_i(\delta_S) = 0$, for each $i \notin S^+$. This shows that

$$\Phi_i(v) = \sum_{\{S \in \mathscr{F} : i \in \text{ex}(S)\}} a_S^i v(S) + \sum_{\{S \in \mathscr{F} : i \in S^*\}} a_S^i v(S).$$

Since $i \in \text{ex}(S) \iff S \setminus i \in \mathscr{F} \iff S = T \cup i$, where $T \in \mathscr{F}$ and $i \in T^*$, we have

$$\sum_{\{S \in \mathscr{F} : i \in \text{ex}(S)\}} a_S^i v(S) = \sum_{\{T \in \mathscr{F} : i \in T^*\}} a_{T \cup i}^i v(T \cup i).$$

If $i \in S^*$ then player i is dummy in the game $\delta_S + \delta_{S \cup i}$ by Lemma 12 (iii). By linearity and dummy axioms

$$a_S^i + a_{S \cup i}^i = \Phi_i(\delta_S) + \Phi_i(\delta_{S \cup i}) = \Phi_i(\delta_S + \delta_{S \cup i}) = 0,$$

which implies that $a_S^i = -a_{S \cup i}^i$ for all $i \in S^*$. Then the above properties yield

$$\Phi_i(v) = \sum_{\{T\in\mathscr{F}: i\in T^*\}} a^i_{T\cup i}[v(T\cup i) - v(T)].$$

Now we suppose that $\{i\}\in\mathscr{F}$ and compute

$$\sum_{\{T\in\mathscr{F}: i\in T^*\}} a^i_{T\cup i} = \sum_{\{T\in\mathscr{F}: i\in T^*\}} \Phi_i(\delta_{T\cup i}) = \Phi_i\left(\sum_{\{T\in\mathscr{F}: i\in T^*\}} \delta_{T\cup i}\right).$$

We claim that player i is dummy in the game $w = \sum_{\{T\in\mathscr{F}: i\in T^*\}}\delta_{T\cup i}$. Observing that for any $C\in\mathscr{F}$ such that $i\in C^*$ the feasible set $T\cup i\neq C$, we obtain

$$\delta_{T\cup i}(C\cup i) - \delta_{T\cup i}(C) = \begin{cases} 1 & \text{if } C = T,\\ 0 & \text{if } C\neq T.\end{cases}$$

This implies that $w(C\cup i) - w(C) = 1$. Observe that $\{i\}\in\mathscr{F}$ implies $i\in\emptyset^+$, and hence $w(\{i\}) = \delta_{\emptyset\cup i}(\{i\}) = 1$. This proves the claim. Finally, by using dummy axiom, we get $\Phi_i(w) = w(\{i\}) = 1$. □

If the vector $\Phi(v) = (\Phi_1(v),\ldots,\Phi_n(v))$ is a distribution of the available resources to the grand coalition $N\in\mathscr{F}$, then Φ satisfies the following axiom:

Efficiency axiom: If $(N,\mathscr{F})$ is an augmenting system such that $N\in\mathscr{F}$ and $v\in\Gamma(\mathscr{F})$ then $\sum_{i\in N}\Phi_i(v) = v(N)$.

The efficiency axiom implies the following properties for the coefficients of the values that satisfy linearity an dummy axioms. We will assume throughout that $(N,\mathscr{F})$ is an augmenting system such that $N\in\mathscr{F}$.

Theorem 14. *Let $\Phi:\Gamma(\mathscr{F})\to\mathbb{R}^n$ be a value defined for every game $v\in\Gamma(\mathscr{F})$ and every player $i\in N$ by*

$$\Phi_i(v) = \sum_{\{S\in\mathscr{F}: i\in S^*\}} a^i_{S\cup i}[v(S\cup i) - v(S)].$$

Then Φ satisfies the efficiency axiom if and only if

$$\sum_{i\in \mathrm{ex}(N)} a^i_N = 1, \quad \text{and} \quad \sum_{i\in \mathrm{ex}(S)} a^i_S = \sum_{i\in S^*} a^i_{S\cup i}$$

for every nonempty $S\in\mathscr{F}$ such that $S\neq N$.

Proof. First, we compute the sum

$$\begin{aligned}\sum_{i\in N}\Phi_i(v) &= \sum_{i\in N}\sum_{\{S\in\mathscr{F}: i\in S^*\}} a^i_{S\cup i}[v(S\cup i) - v(S)]\\ &= \sum_{S\in\mathscr{F}} v(S)\left(\sum_{i\in\mathrm{ex}(S)} a^i_S - \sum_{i\in S^*} a^i_{S\cup i}\right)\\ &= \left(\sum_{i\in\mathrm{ex}(N)} a^i_N\right)v(N) + \sum_{\{S\in\mathscr{F}: S\neq N\}}\left(\sum_{i\in\mathrm{ex}(S)} a^i_S - \sum_{i\in S^*} a^i_{S\cup i}\right)v(S).\end{aligned}$$

By considering $v(S)$ as variables, we conclude that $\sum_{i\in N}\Phi_i(v) = v(N)$ if and only if the relations are true. □

Let $v: 2^N\to\mathbb{R}$ be a standard cooperative game and let π a total ordering of the elements of N, given by $i_1 < i_2 < \cdots < i_n$. The classical Shapley value for the player $i\in N$ is given by

$$Sh_i(N,v) = \frac{1}{n!}\sum_{\pi\in\Pi_n}[v(\pi^i\cup\{i\}) - v(\pi^i)],$$

where Π_n is the set of all permutations of N and π^i is the set of the predecessors of player i in the order π.

Let us consider a compatible ordering of an augmenting system $(N,\mathscr{F})$ such that $N\in\mathscr{F}$, as the total ordering of N, given by $i_1 < i_2 < \cdots < i_n$ such that the set $\{i_1,\ldots,i_j\}\in\mathscr{F}$ for all $j = 1,\ldots,n$. A compatible ordering of $(N,\mathscr{F})$ corresponds exactly to a maximal chain in $\mathscr{F}$ and we denote by $\mathrm{Ch}(\mathscr{F})$ the set of all the maximal chains in $\mathscr{F}$. Given an element $i\in N$ and a compatible ordering $C\in\mathrm{Ch}(\mathscr{F})$, let $C(i) = \{j\in N : j\leqslant i \text{ in } C\}$.

Let $(N,\mathscr{F})$ be an augmenting system and let $v:\mathscr{F}\to\mathbb{R}$ a cooperative game. We define the Shapley value for the player $i\in N$ as

$$Sh_i(N,v,\mathscr{F}) := \frac{1}{c(N)}\sum_{C\in\mathrm{Ch}(\mathscr{F})}[v(C(i)) - v(C(i)\setminus i)],$$

where $c(N) := |\mathrm{Ch}(\mathscr{F})|$ is the total number of maximal chains in $\mathscr{F}$. Since $C(i)\setminus i = S\in\mathscr{F}$ we have that $i\in S^*$. Thus

$$\begin{aligned}\mathrm{Sh}_i(N,v,\mathscr{F}) &= \sum_{\{S\in\mathscr{F}: i\in S^*\}}\left(\sum_{\{C\in\mathrm{Ch}(\mathscr{F}): C(i)\setminus i = S\}}\frac{1}{c(N)}\right)[v(S\cup i) - v(S)]\\ &= \sum_{\{S\in\mathscr{F}: i\in S^*\}}\frac{c(S)c(S\cup i,N)}{c(N)}[v(S\cup i) - v(S)],\end{aligned}$$

where $c(S)$ is the number of maximal chains from $\emptyset$ to S, and $c(S\cup i,N)$ is the number of maximal chains from $S\cup i$ to N.

As a consequence, we obtain the following formula for the Shapley value of games on augmenting systems.

Definition 15. Let $v:\mathscr{F}\to\mathbb{R}$ be a game on an augmenting system $(N,\mathscr{F})$ such that $N\in\mathscr{F}$. The Shapley value for the player $i\in N$ is given by

$$Sh_i(N,v,\mathscr{F}) = \sum_{\{S\in\mathscr{F}: i\in S^*\}}\frac{c(S)c(S\cup i,N)}{c(N)}[v(S\cup i) - v(S)].$$

Note that the sum of the coefficients of the Shapley value is

$$\sum_{\{S\in\mathscr{F}: i\in S^*\}}\left(\sum_{\{C\in\mathrm{Ch}(\mathscr{F}): C(i)\setminus i = S\}}\frac{1}{c(N)}\right) = \sum_{C\in\mathrm{Ch}(\mathscr{F})}\frac{1}{c(N)} = 1$$

and this implies that the Shapley value satisfies the dummy axiom.

Moreover, for every game $v\in\Gamma(\mathscr{F})$ we have

$$\begin{aligned}\sum_{i\in N} Sh_i(N,v,\mathscr{F}) &= \sum_{i\in N}\left(\frac{1}{c(N)}\sum_{C\in\mathrm{Ch}(\mathscr{F})}[v(C(i)) - v(C(i)\setminus i)]\right)\\ &= \frac{1}{c(N)}\sum_{C\in\mathrm{Ch}(\mathscr{F})}\left(\sum_{i\in N}[v(C(i)) - v(C(i)\setminus i)]\right)\\ &= \frac{1}{c(N)}\sum_{C\in Ch(\mathscr{F})}[v(N) - v(\emptyset)] = v(N),\end{aligned}$$

which implies the efficiency axiom.

Since the classical axiom of symmetry does not work, we consider a new axiom in which there is a relationship between the number of chains and the value of the identity game.

Chain axiom: Let $(N,\mathscr{F})$ be an augmenting system such that $N\in\mathscr{F}$ and $\Phi:\Gamma(\mathscr{F})\to\mathbb{R}^n$ a value. For any $S\in\mathscr{F}$ such that $S\neq N$ and any $i,j\in S^*$, we have $c(S\cup i,N)\Phi_j(\delta_{S\cup j}) = c(S\cup j,N)\Phi_i(\delta_{S\cup i})$.

Combining this axiom with the efficiency axiom, we obtain the probability that a player joins coalition $S\in\mathscr{F}$ over the set $\mathrm{Ch}(\mathscr{F})$ of all the maximal chains in $\mathscr{F}$.

By using the previous results we prove the following characterization of the Shapley value for games on augmenting systems.

Theorem 16. *The Shapley value is the unique value $\Phi:\Gamma(\mathscr{F})\to\mathbb{R}^n$ that satisfies linearity, dummy, efficiency and chain axioms.*

Proof. Clearly the Shapley value satisfies all the four axioms. Conversely, let Φ be a value that satisfies linearity, dummy, efficiency and chain axioms. It follows from Theorems 13 and 14 that for every game $v\in\Gamma(\mathscr{F})$ and every player $i\in N$

$$\Phi_i(v) = \sum_{\{S\in\mathscr{F}: i\in S^*\}} a^i_{S\cup i}[v(S\cup i) - v(S)],$$

where the coefficients satisfy

$$\sum_{i\in \mathrm{ex}(N)} a_N^i = 1, \quad \text{and} \quad \sum_{i\in \mathrm{ex}(S)} a_S^i = \sum_{i\in S^*} a_{S\cup i}^i$$

for every nonempty $S \in \mathscr{F}$ such that $S \neq N$. Thus, it suffices to show that

$$a_{S\cup i}^i = \frac{c(S)c(S\cup i, N)}{c(N)} \tag{1}$$

for every $S \in \mathscr{F}$ such that $S\neq N$ and $i \in S^*$. Note that the chain axiom is

$$c(S\cup i, N)a_{S\cup j}^j = c(S\cup j, N)a_{S\cup i}^i$$

for all $i, j \in S^*$. Let us consider a fix coalition $S \in \mathscr{F}$ such that $S\neq N$ with $i \in S^*$, and we compute

$$\begin{aligned}\sum_{j\in S^*} a_{S\cup j}^j &= a_{S\cup i}^i + \sum_{\{j\in S^*:j\neq i\}} \frac{c(S\cup j, N)}{c(S\cup i, N)} a_{S\cup i}^i \\ &= \frac{a_{S\cup i}^i}{c(S\cup i, N)}\left[c(S\cup i, N) + \sum_{\{j\in S^*:j\neq i\}} c(S\cup j, N)\right] \\ &= a_{S\cup i}^i \frac{c(S,N)}{c(S\cup i, N)}.\end{aligned}$$

For $S = \emptyset$ the above equality is

$$\sum_{j\in \emptyset^*} a_{\{j\}}^j = a_{\{i\}}^i \frac{c(N)}{c(\{i\}, N)},$$

where $\{j \in N : j \in \emptyset^*\} = \{j \in N : \{j\} \in \mathscr{F}\}$. By using recursively the efficiency equations

$$\sum_{i\in \mathrm{ex}(S)} a_S^i = \sum_{i\in S^*} a_{S\cup i}^i$$

and the equivalence $i \in S^* \Longleftrightarrow i \in \mathrm{ex}(S\cup i)$, we calculate the sum

$$\begin{aligned}\sum_{j\in\emptyset^*} a_{\{j\}}^j &= \sum_{j\in \mathrm{ex}(\{j\})} a_{\{j\}}^j = \sum_{\{S\in\mathscr{F}:|S|=1\}} \sum_{i\in S^*} a_{S\cup i}^i \\ &= \sum_{\{S\in\mathscr{F}:|S|=2\}} \sum_{i\in \mathrm{ex}(S)} a_S^i = \sum_{\{S\in\mathscr{F}:|S|=2\}} \sum_{i\in S^*} a_{S\cup i}^i \\ &\ \ \vdots \\ &= \sum_{\{S\in\mathscr{F}:|S|=n-1\}} \sum_{i\in \mathrm{ex}(S)} a_S^i = \sum_{\{S\in\mathscr{F}:|S|=n-1\}} \sum_{i\in S^*} a_{S\cup i}^i \\ &= \sum_{i\in \mathrm{ex}(N)} a_N^i = 1.\end{aligned}$$

Therefore, we have showed formula (1) for $S = \emptyset$, that is, for every $i \in \emptyset^*$,

$$a_{\{i\}}^i = \frac{c(\{i\}, N)}{c(N)}.$$

We assume the following induction hypothesis: For every $T \in \mathscr{F}$ such that $|T| = k$, where $0 \leqslant k \leqslant n-2$, we have

$$a_{T\cup j}^j = \frac{c(T)c(T\cup j, N)}{c(N)}$$

for all $j \in T^*$. The case $|T| = 0 \Longleftrightarrow T = \emptyset$ has just been proved. Let now $S \in \mathscr{F}$ such that $|S| = k+1 \leqslant n-1$. Then $\emptyset\neq S\neq N$, and hence the efficiency equations imply that

$$\sum_{j\in S^*} a_{S\cup j}^j = \sum_{j\in \mathrm{ex}(S)} a_S^j = \sum_{j\in \mathrm{ex}(S)} a_{(S\setminus j)\cup j}^j = \sum_{j\in \mathrm{ex}(S)} \frac{c(S\setminus j)c(S,N)}{c(N)} = \frac{c(S)c(S,N)}{c(N)},$$

where we have used the induction hypothesis for $T = S\setminus j$ where $j \in \mathrm{ex}(S)$. Moreover

$$\sum_{j\in S^*} a_{S\cup j}^j = a_{S\cup i}^i \frac{c(S,N)}{c(S\cup i, N)},$$

which implies the formula (1) for $S \in \mathscr{F}$ such that $|S| = k+1$. This proves that $\Phi_i(N, v, \mathscr{F}) = Sh_i(N, v, \mathscr{F})$ for every $v \in \Gamma(\mathscr{F})$ and every $i \in N$. □

Remark 17. Note that if $\mathscr{F} = 2^N$ then $\{S \in \mathscr{F} : i \in S^*\} = \{S \subset N : i \notin S\}$. Thus, for every game $v : 2^N \to \mathbb{R}$ and every $i \in N$, we have

$$Sh_i(N, v) = \sum_{\{S\subset N: i\notin S\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}[v(S\cup i) - v(S)].$$

4. Another axiomatization of the Shapley value

Given an augmenting system $(N, \mathscr{F})$ we consider the partially ordered set (or poset) $(\mathscr{F}, \subseteq)$. Let us denote by $\mathrm{Int}(\mathscr{F})$ the set of intervals of $(N, \mathscr{F})$, that is the collections $[S,T] = \{R \in \mathscr{F} : S\subseteq R\subseteq T\}$, where $S, T \in \mathscr{F}$ and $S\subseteq T$. We define the zeta function $\zeta : \mathrm{Int}(\mathscr{F}) \to \mathbb{R}$ by $\zeta(S,T) = 1$ for all $S, T \in \mathscr{F}$ such that $S\subseteq T$. The identity function $\delta : \mathrm{Int}(\mathscr{F}) \to \mathbb{R}$ is defined by $\delta(S,T) = 1$ if $S = T$ and $\delta(S,T) = 0$ otherwise. The convolution of the functions $f, g : \mathrm{Int}(\mathscr{F}) \to \mathbb{R}$ is

$$(f*g)(S,T) = \sum_{\{R\in\mathscr{F}:S\subseteq R\subseteq T\}} f(S,R)g(R,T)$$

and the identity function satisfies $f*\delta = \delta*f = f$. Moreover, the zeta function ζ is invertible, its inverse is called the *Möbius function* and is denoted μ (see [17, Section 3.7]).

Lemma 18. *Let $(N, \mathscr{F})$ be an augmenting system. Then the Möbius function of the poset $(\mathscr{F}, \subseteq)$ is*

$$\mu(S,T) = \begin{cases}(-1)^{|T|-|S|} & \text{if } T\subseteq S^+, \\ 0 & \text{otherwise}.\end{cases}$$

Proof. It suffices to show that

$$(\mu*\zeta)(S,T) = \delta(S,T) = \begin{cases}1 & \text{if } S = T, \\ 0 & \text{if } S \subset T,\end{cases}$$

for all $S, T \in \mathscr{F}$ such that $S\subseteq T$. By Proposition 7 we obtain

$$\begin{aligned}(\mu*\zeta)(S,T) &= \sum_{\{R\in\mathscr{F}:S\subseteq R\subseteq T\}} \mu(S,R)\zeta(R,T) = \sum_{\{R\in\mathscr{F}:S\subseteq R\subseteq T\}} \mu(S,R) \\ &= \sum_{\{R\in 2^N:S\subseteq R\subseteq S^+, R\subseteq T\}} (-1)^{|R|-|S|}.\end{aligned}$$

If $S = T$ then $R = S$ and hence $(\mu*\zeta)(S,T) = (-1)^{|S|-|S|} = 1$. If $S \subset T$ then we consider two cases:

1. Assume $S^+ \subseteq T$ and let $C = R\setminus S$. Then

$$\begin{aligned}(\mu*\zeta)(S,T) &= \sum_{\{R\in 2^N:S\subseteq R\subseteq S^+\}} (-1)^{|R|-|S|} = \sum_{\{C\in 2^N:C\subseteq S^+\setminus S\}} (-1)^{|C|} \\ &= (1-1)^{|S^+\setminus S|} = \begin{cases}1 & \text{if } S = S^+, \\ 0 & \text{otherwise}.\end{cases}\end{aligned}$$

   Since $S = S^+$ implies that $S = N$, we obtain a contradiction with $S \subset T$, and hence $(\mu*\zeta)(S,T) = 0$.
2. Assume $S^+ \nsubseteq T$ and let $S_T^* = \{i \in T\setminus S : S\cup i \in \mathscr{F}\}$. Then

$$\{R \in 2^N : S\subseteq R\subseteq S^+, R\subseteq T\} = \{R \in 2^N : S\subseteq R\subseteq S\cup S_T^*\}.$$

   We now obtain

$$(\mu * \zeta)(S,T) = \sum_{\{R \in 2^N : S \subseteq R \subseteq S \cup S_T^*\}} (-1)^{|R|-|S|} = (1-1)^{|S_T^*|} = \begin{cases} 1 & \text{if } S_T^* = \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $S_T^* = \emptyset$ implies that $S \cup i \notin \mathscr{F}$ for any $i \in T \setminus S$. This contradicts property P3 of the augmenting system and we conclude that $(\mu * \zeta)(S,T) = 0$. □

For any $T \in \mathscr{F}$ such that $T \neq \emptyset$, the *unanimity game* $\zeta_T : \mathscr{F} \to \mathbb{R}$ is defined by

$$\zeta_T(S) := \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise}. \end{cases}$$

The collections of the identity games $\{\delta_S : S \in \mathscr{F}, S \neq \emptyset\}$ and the unanimity games $\{\zeta_T : T \in \mathscr{F}, T \neq \emptyset\}$ are two different bases of the vector space $\Gamma(\mathscr{F})$. Faigle and Kern [10] observed that

$$\zeta_T = \sum_{\{S \in \mathscr{F} : S \supseteq T\}} \delta_S.$$

Theorem 19. *Let $v : \mathscr{F} \to \mathbb{R}$ be a game on an augmenting system. Then there exists an unique set of coefficients $\{d_v(T) : T \in \mathscr{F}, T \neq \emptyset\}$ such that $v = \sum_{\{T \in \mathscr{F} : T \neq \emptyset\}} d_v(T)\zeta_T$. Moreover,*

$$d_v(S) = \sum_{\{T \in \mathscr{F} : T \subseteq S \subseteq T^+\}} (-1)^{|S|-|T|} v(T).$$

for every nonempty $S \in \mathscr{F}$.

Proof. The collection of the unanimity games $\{\zeta_T : T \in \mathscr{F}, T \neq \emptyset\}$ is a basis of the vector space $\Gamma(\mathscr{F})$. Then, for every game $v \in \Gamma(\mathscr{F})$

$$v = \sum_{\{T \in \mathscr{F} : T \neq \emptyset\}} d_v(T)\zeta_T$$

and hence for every nonempty $S \in \mathscr{F}$ we have that

$$v(S) = \sum_{\{T \in \mathscr{F} : T \neq \emptyset\}} d_v(T)\zeta_T(S) = \sum_{\{T \in \mathscr{F} : T \subseteq S\}} d_v(T).$$

Applying the Möbius inversion formula [17, Chapter 3] of the poset $(\mathscr{F}, \subseteq)$ and Lemma 18, we obtain

$$d_v(S) = \sum_{\{T \in \mathscr{F} : T \subseteq S\}} \mu(T,S)v(T) = \sum_{\{T \in \mathscr{F} : T \subseteq S \subseteq T^+\}} (-1)^{|S|-|T|} v(T). \quad \square$$

Let $(N, \mathscr{F})$ be an augmenting system such that $N \in \mathscr{F}$. Following the work of Faigle and Kern [10], we define the *hierarchical strength* $h_S(i)$ of a player $i \in S$ in a feasible coalition $S \in \mathscr{F}$ as follows:

$$h_S(i) := \frac{|\{C \in Ch(\mathscr{F}) : C(i) \supseteq S\}|}{c(N)}.$$

Note that $h_S(i)$ is the average number of maximal chains of $(\mathscr{F}, \subseteq)$ in which player $i \in S$ is the last member of S in the chain. By using these numbers we will obtain a new formula for the Shapley value.

Proposition 20. *Let $v : \mathscr{F} \to \mathbb{R}$ be a game on an augmenting system $(N, \mathscr{F})$ such that $N \in \mathscr{F}$. The Shapley value for the player $i \in N$ is given by*

$$Sh_i(N, v, \mathscr{F}) = \sum_{\{S \in \mathscr{F} : i \in S\}} d_v(S)h_S(i),$$

where $d_v(S)$ are the coefficients associated to the unanimity basis.

Proof. Since $v = \sum_{\{S \in \mathscr{F} : S \neq \emptyset\}} d_v(S)\zeta_S$ the linearity of the Shapley value implies that

$$Sh_i(N, v, \mathscr{F}) = \sum_{\{S \in \mathscr{F} : S \neq \emptyset\}} d_v(S) Sh_i(N, \zeta_S, \mathscr{F}).$$

For every nonempty $S \in \mathscr{F}$ and $i \in N$ we compute

$$Sh_i(N, \zeta_S, \mathscr{F}) = \frac{1}{c(N)} \sum_{C \in Ch(\mathscr{F})} [\zeta_S(C(i)) - \zeta_S(C(i) \setminus i)].$$

If $i \notin S$ then $S \subseteq C(i)$ implies $S \subseteq C(i) \setminus i$, and hence

$$\zeta_S(C(i)) - \zeta_S(C(i) \setminus i) = \begin{cases} 1 & \text{if } S \subseteq C(i) \text{ and } i \in S \\ 0 & \text{otherwise} \end{cases}$$

for every chain $C \in Ch(\mathscr{F})$. Thus, we obtain

$$\mathrm{Sh}_i(N, \zeta_S, \mathscr{F}) = \begin{cases} h_S(i) & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases}$$

which completes the proof. □

Now we are ready to introduce a new axiom which gives rise to another axiomatization of the Shapley value.

Hierarchical strength axiom: Let $(N, \mathscr{F})$ be an augmenting system such that $N \in \mathscr{F}$ and $\Phi : \Gamma(\mathscr{F}) \to \mathbb{R}^n$ a value. For any nonempty $S \in \mathscr{F}$ and any $i, j \in S$, we have $h_S(i)\Phi_j(\zeta_S) = h_S(j)\Phi_i(\zeta_S)$.

This axiom implies that players in unanimity games be rewarded according to their relative hierarchical strengths. Moreover, it reflect that the Shapley value is the expected marginal contribution of an individual player to the game. Note also that the above proposition implies that the Shapley value satisfies the hierarchical strength axiom.

Theorem 21. *The Shapley value is the unique value $\Phi : \Gamma(\mathscr{F}) \to \mathbb{R}^n$ that satisfies linearity, dummy, efficiency and hierarchical strength axioms.*

Proof. We know that the Shapley value satisfies the four axioms. Since Φ is a linear map and $v = \sum_{\{T \in \mathscr{F} : T \neq \emptyset\}} d_v(T)\zeta_T$, it suffices to prove that Φ coincides with the Shapley value on any game ζ_T, where $T \in \mathscr{F}$ and $T \neq \emptyset$. Fix a nonempty $T \in \mathscr{F}$ and $i \in N$. We show that any $i \notin T$ is a dummy player in the game ζ_T. For this, let $S \in \mathscr{F}$ such that $i \in S^*$. Since $i \notin T$ we have that $T \subseteq S \cup i$ implies $T \subseteq S$, and hence $\zeta_T(S \cup i) - \zeta_T(S) = 0$. Moreover, if $\{i\} \in \mathscr{F}$ then $\zeta_T(\{i\}) = 0$. By using the dummy axiom we obtain that $\Phi_i(\zeta_T) = 0$ for every $i \notin T$.

By applying the efficiency axiom we have

$$\sum_{i \in T} \Phi_i(\zeta_T) = \sum_{i \in N} \Phi_i(\zeta_T) = \zeta_T(N) = 1.$$

Now we fix $i \in T$ and the hierarchical strength axiom gives

$$\Phi_j(\zeta_T) = \frac{h_T(j)}{h_T(i)} \Phi_i(\zeta_T)$$

for every $j \in T$ such that $j \neq i$. Thus

$$1 = \sum_{i \in T} \Phi_i(\zeta_T) = \Phi_i(\zeta_T) + \sum_{\{j \in T : j \neq i\}} \frac{h_T(j)}{h_T(i)} \Phi_i(\zeta_T) = \frac{\Phi_i(\zeta_T)}{h_T(i)} \sum_{j \in T} h_T(j).$$

Since in every chain $C \in Ch(\mathscr{F})$ there exists an unique $j \in T$ such that $C(j) \supseteq T$, we have

$$\sum_{j \in T} h_T(j) = 1$$

and hence we conclude that $\Phi_i(\zeta_T) = h_T(i)$ for every $i \in T$. Therefore, $\Phi_i(\zeta_T) = Sh_i(\zeta_T)$ for every nonempty $T \in \mathscr{F}$ and $i \in N$. □

Acknowledgements

This research has been partially supported by the Spanish Ministry of Education and Science and the European Regional

Development Fund, under Grant SEJ2006–00706, and by the FQM 237 Grant of the Andalusian Government. The authors are grateful for the constructive and detailed comments and suggestions made by the anonymous reviewers which lead to a considerable improvement of our paper.

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