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\textbf{ABSTRACT}

A nontransferable utility (NTU) game assigns a set of feasible pay-off vectors to each coalition. In this article, we study NTU games in situations in which there are restrictions on coalition formation. These restrictions will be modelled through interior structures, which extend some of the structures considered in the literature on transferable utility games for modelling restricted cooperation, such as permission structures or antimatroids. The Harsanyi value for NTU games is extended to the set of NTU games with interior structure.

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\section{1. Introduction}

Cooperative games are mathematical models which have been proposed to study situations in which a set of players aim to share the profit derived from their cooperation. A cooperative game involves a finite set of players $N$, whose subsets are called coalitions, and a characteristic function, which specifies the outcomes that each coalition can achieve for itself. When a coalition $S \subseteq N$ is formed, each player in $S$ can receive an amount of utility. If for any coalition and any player in the coalition, this player can freely transfer any part of her/his utility to any other player in the coalition then the game is said to be a game with transferable utility or TU game. In the case of a TU game, the characteristic function is a mapping $v : 2^N \rightarrow \mathbb{R}$ that assigns to each coalition the value of the utilities that can be obtained by the coalition. If the utilities cannot be freely transferred among the agents then we need another model. This is the origin of nontransferable utility games or NTU games, which were introduced by Aumann and Peleg \cite{1}. In this case, the characteristic function is a mapping $V$ that assigns to each coalition $S \subseteq N$ the set of all the possible utility distributions for coalition $S$. Since a utility distribution in $S$ is given by a vector in $\mathbb{R}^S$, $V(S)$ is a subset of $\mathbb{R}^S$. Actually, NTU games can be seen as an extension of TU games. NTU games have multiple economic applications.

Given a cooperative game, the main problem that arises is how to distribute the profit that the players obtain from their cooperation. In order to solve this problem, game theorists introduced the solution concepts. A solution concept is a correspondence that assigns a pay-off (or set of pay-offs) to each cooperative game. Many solutions concepts have been defined and studied in the literature. In the case of TU games, the best-known solution concept is the Shapley value. In the case of the pure bargaining problems, which are a family of NTU games, the classical solution is the Nash bargaining solution. This leads to look for solution concepts for NTU games that extend both the Shapley value and the Nash bargaining solution. An NTU value is a solution concept for NTU games that satisfies these two conditions and, in addition, it is covariant with individual pay-off rescalings. The most
notable of the NTU values proposed in the literature are the Harsanyi value [2], the Shapley value [3] and the Maschler–Owen consistent value [4].

Given a cooperative game, it is often assumed that the players are free to participate in any coalition, but in some situations there are dependency relationships among the players that restrict their capacity to play within some coalitions. Those relationships must be taken into account if we want to distribute the profits fairly. This led several authors to develop models of cooperative games with restricted cooperation. Numerous types of restrictions on the cooperation among the players have been studied, and multiple structures have been used, such as antimatroids [5], matroids [6], convex geometries [7], augmenting systems [8] or restrictions [9]. However, they have been applied mainly to TU games. Very few models of restricted cooperation have been proposed for NTU games (see, for instance, Bergantiños and Vidal-Puga [10]). The goal of this paper is to introduce a new model of NTU games with restricted cooperation. In order to model the limitations on cooperation, interior operators will be used. These operators can model different types of dependency relations among the players. We introduce NTU games with interior structure, which consist of a set of players \(N\), an NTU game on \(N\) and an interior operator on \(N\). The problem we address is how to distribute the total profit. To this end, firstly we associate a restricted game to each NTU game with interior structure. This restricted game indicates the outcomes that each coalition can achieve when the limitations on cooperation are taken into account. And secondly, the Harsanyi value will be applied to the restricted game. In this way, a value for NTU games with interior structure will be obtained.

The paper is organized as follows. In Section 2, several basic definitions and results concerning cooperative games are recalled. Interior operators are introduced in Section 3. In Section 4, a value for NTU games with interior structure is defined and characterized. In Section 5, some conclusions are drawn.

2. Preliminaries

2.1. Transferable utility games

A transferable utility game or TU game is a pair \((N, v)\), where \(N\) is a set of cardinality \(n\) with \(n \in \mathbb{N}\) and \(v : 2^N \to \mathbb{R}\) is a function that satisfies \(v(\emptyset) = 0\). The elements of \(N\) are called players, the subsets \(E \subseteq N\) are called coalitions and the number \(v(E)\) is the worth of \(E\). Often, the TU game \((N, v)\) is identified with the function \(v\). If \(v(E) \leq v(F)\) for all \(E \subseteq F \subseteq N\) then the game \(v\) is said to be monotonic. The set of all TU games on \(N\) is denoted by \(\mathcal{G}^N\). If \(v \in \mathcal{G}^N\) and \(E \in 2^N \setminus \{\emptyset\}\) then \(v|_E\) denotes the restriction of \(v\) to \(2^E\).

Given a TU game \((N, v)\), a problem that arises is how to assign a pay-off to each player in a fair way. An allocation rule or value assigns to each game \((N, v)\) a Pay-off vector \(\psi(v) \in \mathbb{R}^N\). Several allocation rules have been defined in the literature. The most notable of them is the Shapley value, introduced by Shapley [11] in 1953. Given \(v \in \mathcal{G}^N\), the Shapley value of \(v\), denoted by \(\phi(v)\), is defined as

\[
\phi_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E \left[v(E) - v(E \setminus \{i\})\right] \quad \text{for all } i \in N,
\]

where \(p_E = \frac{(n - |E|)!}{n!} \frac{(|E| - 1)!}{|E|!} \) and \(|E|\) denotes the cardinality of \(E\).

2.2. Nontransferable utility games

A cooperative game with nontransferable utility or NTU game is a pair \((N, V)\) where \(N\) is a set of cardinality \(n \in \mathbb{N}\) and \(V\) is a correspondence that assigns to each nonempty \(E \subseteq N\) a nonempty subset \(V(E) \subseteq \mathbb{R}^E\). The set valued function \(V\) is the characteristic function of the NTU game \((N, V)\). Often, the NTU game \((N, V)\) is identified with the function \(V\).
If $V$ and $W$ are NTU games, the NTU game $V + W$ is defined by

$$(V + W)(E) = V(E) + W(E) = \{x + y : x \in V(E), y \in W(E)\}$$

for every $E \in 2^N \setminus \{\emptyset\}$. If $\alpha \in \mathbb{R}^N$, the NTU game $\alpha \ast V$ is defined by $(\alpha \ast V)(E) = \alpha^E \ast V(E)$ for every $E \in 2^N \setminus \{\emptyset\}$ where $\alpha^E$ is the restriction of $\alpha$ to $E$ (i.e. $\alpha^E \in \mathbb{R}^E$ and $\alpha_i^E = \alpha_i$ for every $i \in E$) and $\ast$ denotes the Hadamard product. Given $v \in \mathcal{G}^N$ the NTU game corresponding to $v$ is defined as

$$V_v(E) = \left\{ y \in 0^E : \sum_{k \in E} y_k \leq v(E) \right\} \text{ for every } E \in 2^N \setminus \{\emptyset\}.$$  

The Harsanyi configuration correspondence for NTU games was introduced by Harsanyi [2] and it was characterized by Hart [12]. Hart considered the NTU games $V$ satisfying the following conditions:

(i) $V(E)$ is closed, convex and comprehensive for every $E \in 2^N \setminus \{\emptyset\}$.  
(ii) $V(N)$ is smooth. We recall that a convex subset $C$ of $\mathbb{R}^N$ is smooth if it has a unique supporting hyperplane at each point of its boundary $\partial C$.  
(iii) For every $x \in \partial V(N)$, $\{y \in \mathbb{R}^N : y \geq x\} \cap V(N) = \{x\}$.  

We denote $\Omega^N$ the set of NTU games satisfying (i), (ii) and (iii).  

A pay-off configuration for $N$ is an element $(x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \prod_{E \in 2^N \setminus \{\emptyset\}} 0^E$. Let $V \in \Omega^N$. A pay-off configuration $(x^E)_{E \in 2^N \setminus \{\emptyset\}}$ is a Harsanyi solution of $V$ if there exists $\lambda \in \mathbb{R}^N_{++}$ such that

1. $x^E \in \partial (V(E))$ for every $E \in 2^N \setminus \{\emptyset\}$,  
2. $\lambda \cdot x^N = \max \{\lambda \cdot z : z \in V(N)\}$,  
3. $\lambda^E \ast x^E = \phi(w|E)$ for every $E \in 2^N \setminus \{\emptyset\}$, where $w$ is the TU game given by $w(F) = \lambda^F \cdot x^F$ for every $F \in 2^N \setminus \{\emptyset\}$.  

The mapping that assigns to each $V \in \Omega^N$ the set of Harsanyi solutions of $V$ is called the Harsanyi configuration correspondence for NTU games (on $N$) and is denoted by $H$. Hart [12] proved that $H$ satisfies the following properties:  

CONDITIONAL ADDITIVITY. If $V, W \in \Omega^N$ are such that $V + W = U \in \Omega^N$ and $(x^E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V) + H(W)$ are such that $x^E \in \partial (U(E))$ for every $E \in 2^N \setminus \{\emptyset\}$, then $(x^E)_{E \in 2^N \setminus \{\emptyset\}} \in H(U)$.  

SCALE COVARIANCE. If $V \in \Omega^N$ and $\alpha \in \mathbb{R}^N_{++}$, then  

$$H(\alpha \ast V) = \left\{(\alpha^E \ast x^E)_{E \in 2^N \setminus \{\emptyset\}} : (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V)\right\}.$$  

INDEPENDENCE OF IRRELEVANT ALTERNATIVES. If $V, W \in \Omega^N$ are such that $V(E) \subseteq W(E)$ for every $E \in 2^N \setminus \{\emptyset\}$ and $(x^E)_{E \in 2^N \setminus \{\emptyset\}} \in H(W)$ are such that $x^E \in V(E)$ for every $E \in 2^N \setminus \{\emptyset\}$, then $(x^E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V)$.  

3. Games with interior structure  

In order to model situations with restricted cooperation, the concept of interior operator is introduced.  

**Definition 3.1:** An interior operator on $N$ is a mapping $A : 2^N \to 2^N$ that satisfies the following conditions:
(1) \( A(E) \subseteq E \) for any \( E \subseteq N \),
(2) If \( E \subseteq F \) then \( A(E) \subseteq A(F) \),
(3) \( A(N) = N \),
(4) \( A(A(E)) = A(E) \) for every \( E \subseteq N \).

The pair \((N, A)\) is called an interior structure.

If \( A \) is an interior operator on \( N \) and \( E \in 2^N \setminus \{\emptyset\} \) then \( A|E \) denotes the restriction of \( A \) to \( 2^E \).

If \( A \) is an interior operator on \( N \) and \( E \subseteq N \), \( A(E) \) can be interpreted as the set of players that will be allowed to play when coalition \( E \) is formed. Let us give an example.

**Example 3.2:** Let \( N = \{1, 2, 3\} \). Suppose that 3 is subordinate to players 1 and 2, that is, 3 cannot play within a coalition that does not contain players 1 and 2. The following digraph represents these veto relations:

![Veto Relations Digraph]

Let us model this situation of restricted cooperation by means of an interior operator. Let \( A \) be the interior operator on \( N \) defined as

\[
A(E) = \begin{cases} 
E \setminus \{3\} & \text{if } E \neq \{1, 2, 3\}, \\
\{1, 2, 3\} & \text{if } E = \{1, 2, 3\}.
\end{cases}
\]

When a coalition \( E \subseteq N \) is formed, only the players in \( A(E) \) are allowed to play. For instance, if coalition \( \{1, 3\} \) is formed, only 1 will be able to play. Agent 3 could not play because of the veto of 2 over 3. Therefore, \( A(\{1, 3\}) = \{1\} \).

Interior operators are particular cases of the restrictions introduced by Derks and Peters [9]. In fact, the only difference between both concepts is that in the case of restrictions the property \( A(N) = N \) is not required. But, actually, the model proposed by Derks and Peters would not lose generality if they considered just interior operators. Indeed, in their model, given a restriction \( A \) and a game \( v \) on \( N \), the players in \( N \setminus A(N) \) are irrelevant and receive zero pay-off, whereas the players in \( A(N) \) obtain the same pay-off that they would receive if the players in \( N \setminus A(N) \) were eliminated from the game.

Given an interior operator \( A \), we can consider the family

\[
O_A = \{ E \subseteq N : A(E) = E \}
\]

. In this way, we can identify an interior operator with a family of coalitions that is union-closed and contains the empty set and the grand coalition. Via this identification, we can say that some of the families of feasible coalitions considered in the literature to model games with restricted cooperation are interior operators.

The coalitions in \( O_A \setminus \{\emptyset\} \) can be seen as the autonomous coalitions in \((N, A)\), in the sense that if one of these coalitions is formed, all the players in the coalitions will be able to cooperate. We use the notation

\[
\text{aut}(A) = O_A \setminus \{\emptyset\} = \{ E \in 2^N \setminus \{\emptyset\} : A(E) = E \}.
\]
**Definition 3.3:** A game with interior structure on $N$ is a pair $(v, A)$ where $v \in G^N$ and $A$ is an interior operator on $N$.

Given a game with interior structure, a characteristic function that gathers the information from the game and the structure in a reasonable way is defined.

**Definition 3.4:** Let $v \in G^N$ and let $A$ be an interior operator on $N$. The restricted game of $(v, A)$ is the game $v^A \in G^N$ defined as

$$v^A(E) = v(A(E)) \text{ for all } E \subseteq N.$$ 

An allocation rule for games with interior structure assigns to every game with interior structure a pay-off vector.

**Definition 3.5:** The Shapley interior value is defined as

$$\Phi(v, A) = \phi(v^A) \text{ for every } v \in G^N \text{ and every } A \text{ operator interior on } N.$$ 

This value coincides with the Shapley restricted value studied and characterized by Derks and Peters [9]. Another characterization can be found in Gallardo [13].

### 4. The Harsanyi configuration correspondence for NTU games with interior structure

We aim to define and characterize a Harsanyi solution for NTU games with interior structure. In this section, we only consider the NTU games $V \in \Omega^N$.

**Definition 4.1:** An NTU game with interior structure is a pair $(V, A)$ where $V$ is an NTU game on $N$ and $A$ is an interior operator on $N$.

Given an NTU game with interior structure, we define an NTU game that gathers the information from both the game and the structure.

**Definition 4.2:** Let $V$ be an NTU game on $N$ and let $A$ be an interior operator on $N$. The restricted game of $(V, A)$ is the NTU game $V^A$ defined as

$$V^A(E) = \begin{cases} V(E) & \text{if } A(E) = E, \\ (-\infty, 0]^E & \text{if } A(E) = \emptyset, \\ V(A(E)) \times (-\infty, 0]^{E \setminus A(E)} & \text{otherwise,} \end{cases} \text{ for every } E \in 2^N \setminus \{\emptyset\}. $$

The following proposition can be easily verified.

**Proposition 4.3:** Let $V \in \Omega^N$ and let $A$ be an interior operator on $N$. Then $V^A \in \Omega^N$.

A configuration correspondence $\Psi$ for NTU games with interior structure assigns to each $V \in \Omega^N$ and $A$ interior operator on $N$ a subset $\Psi(V, A)$ of $\prod_{E \in 2^N \setminus \{\emptyset\}} \mathbb{R}^E$. We aim to define a configuration correspondence for NTU games with interior structure with certain desirable properties, as we show below.

**Definition 4.4:** The Harsanyi configuration correspondence for NTU games with interior structure, denoted by $\mathcal{H}$, is defined as

$$\mathcal{H}(V, A) = H(V^A) \text{ for every } V \in \Omega^N \text{ and every } \text{ interior operator } A \text{ on } N,$$

where $H$ denotes the Harsanyi configuration correspondence for NTU games.

Bearing in mind the definition of the Harsanyi configuration correspondence for NTU games, we can rewrite the definition of the Harsanyi configuration correspondence for NTU games with interior structure.
Remark 1: Let \( V \in \Omega^N \) and let \( A \) be an interior operator on \( N \). A pay-off configuration \( (x^E)_{E \in 2^N \setminus \{\emptyset\}} \) belongs to \( \mathcal{H}(V, A) \) if there exists \( \lambda \in \mathbb{R}^N_{++} \) such that

1. \( x^E \in \partial \left( V(A)(E) \right) \) for all \( E \in 2^N \setminus \{\emptyset\} \),
2. \( \lambda \cdot x^N = \max \left\{ \lambda \cdot y : y \in V(N) \right\} \),
3. if \( w \) is the TU game given by \( w(F) = \lambda^F \cdot x^F \) for every \( F \in 2^N \setminus \{\emptyset\} \), then \( \lambda^E \cdot x^E = \phi(w|_E) \) for every \( E \in 2^N \setminus \{\emptyset\} \).

We aim to give a characterization of the Harsanyi configuration correspondence for NTU games with interior structure. To this end, we consider the properties stated below. In the statement of these properties, \( \Psi \) is a configuration correspondence for NTU games with interior structure.

- **Efficiency.** If \( V \in \Omega^N \), \( A \) is an interior operator on \( N \) and \( (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A) \), then
  \[
  x^E \in \partial \left( V(E) \right) \quad \text{for all } E \in aut(A).
  \]

- **Conditional additivity.** Let \( V, W \in \Omega^N \) be such that \( V + W \in \Omega^N \), let \( A \) be an interior operator on \( N \) and let \( (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A) + \Psi(W, A) \) be such that \( x^E \in \partial((V + W)(E)) \) for all \( E \in aut(A) \). Then \( (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V + W, A) \).

- **Scale covariance.** If \( V \in \Omega^N \), \( A \) is an interior operator on \( N \) and \( \alpha \in \mathbb{R}^N_{++} \), then
  \[
  \Psi(\alpha \cdot V, A) = \left\{ (\alpha^E \cdot x^E)_{E \in 2^N \setminus \{\emptyset\}} : (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A) \right\}.
  \]

- **Independence of irrelevant alternatives.** Let \( V, W \in \Omega^N \). Let \( A \) be an interior operator on \( N \) such that \( V(E) \subseteq W(E) \) for all \( E \in aut(A) \) and let \( (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(W, A) \) be such that \( x^E \in V(E) \) for all \( E \in aut(A) \). Then
  \[
  (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A) \).
  \]

- **Consistency with the Shapley interior value.** If \( \nu \in G^N \) and \( A \) is an interior operator on \( N \), then
  \[
  \Psi(V, \nu, A) = \left\{ (x^E)_{E \in 2^N \setminus \{\emptyset\}} \right\}
  \]
  where
  \[
  x^E_i = \begin{cases} 
  \Phi_i\left(\nu|_{A(E)}, A_i|_{A(E)}\right) & \text{if } i \in A(E), \\
  0 & \text{if } i \in E \setminus A(E),
  \end{cases}
  \]
  for every \( E \in 2^N \setminus \{\emptyset\} \) and every \( i \in E \).

- **Zero inessential games.** If \( V \in \Omega^N \), \( A \) is an interior operator on \( N \) and \( 0 \in \partial(V(E)) \) for all \( E \in aut(A) \), then
  \[
  (0)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A).
  \]

In the next theorem, it is proved that these properties uniquely determine the Harsanyi configuration correspondence for NTU games with interior structure. First, we show a proposition that will be useful in the proof of the theorem.

**Proposition 4.5:** If \( V \in \Omega^N \), \( A \) is an interior operator on \( N \) and \( (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(V, A) \), then

(a) \( x^F = 0 \) for every \( F \in 2^N \setminus \{\emptyset\} \) with \( A(F) = \emptyset \).
(b) For every \( F \in 2^N \) with \( A(F) \neq \emptyset \),
  \[
  x^F_i = \begin{cases} 
  x^A(F)_i & \text{if } i \in A(F), \\
  0 & \text{if } i \in F \setminus A(F).
  \end{cases}
  \]
**Proof:** Let $V \in \Omega^N$, $A$ an interior operator on $N$ and $(x^E)_{E \in 2^N \setminus \emptyset} \in \mathcal{H}(V, A)$.

(a) We proceed by strong induction on $|F|$.

1. **Base case.** Suppose $|F| = 1$. Then $F = \{i\}$ and $V^A(F) = (-\infty, 0]^i$. Since $x^F \in \partial (V^A(F))$ it must be $x^F = 0$.

2. **Inductive step.** Suppose $|F| > 1$. By induction hypothesis, we have

   $$x^D = 0 \quad \text{for all } D \subsetneq F \text{ with } D \neq \emptyset.$$  

   Let $\lambda \in \mathbb{N}_+^N$, the vector associated to the pay-off configuration $(x^E)_{E \in 2^N \setminus \emptyset}$. From condition (3) of the definition of $\mathcal{H}$ in Remark 1 we obtain that

   $$\lambda^F \cdot x^F = \phi(w_F),$$

   where $w(D) = \lambda^D \cdot x^D$ for all $D \in 2^N \setminus \emptyset$. From (1) it is known that $w(D) = 0$ for all $D \subsetneq F$. Therefore,

   $$\phi_i(w_F) = \frac{w(F)}{|F|} \quad \text{for every } i \in F,$$

   and from (2) and (3) we conclude that

   $$x^F_i = \frac{w(F)}{|F| \lambda_i} \quad \text{for every } i \in F.$$  

   But from condition (1) in Remark 1 we know that $x^F \in \partial (V^A(F)) = \partial ((-\infty, 0]^F)$. It follows that there exists $j \in F$ such that $x^F_j = 0$. Thus $w(F) = 0$. From this and (4), we conclude that $x^F = 0$.

(b) We proceed by strong induction on $|F|$.

1. **Base case.** If $|F| = 1$ there is nothing to prove.

2. **Inductive step.** Suppose $|F| > 1$. We can assume that $A(F) \neq F$ because otherwise there is nothing to prove. Let $\lambda \in \mathbb{N}_+^N$ the vector associated to the pay-off configuration $(x^E)_{E \in 2^N \setminus \emptyset}$. Let $w$ be the TU game given by $w(E) = \lambda^E \cdot x^E$ for every $E \in 2^N \setminus \emptyset$. From (a) and the induction hypothesis we can easily derive that

   $$w(D) = w(A(D)) \quad \text{for every } D \subsetneq F.$$  

   Notice that if $D \subseteq F$ then $A(D) = A(A(D)) \subseteq A(D \cap A(F))$. Therefore, it is clear that

   $$A(D) = A(D \cap A(F)) \quad \text{for every } D \subseteq F.$$  

   From (5) and (6), we conclude that

   $$w(D) = w(D \cap A(F)) \quad \text{for every } D \subsetneq F.$$  

   For each nonempty $D \subseteq F$, we define

   $$u(D) = w(D \cap A(F)).$$

   It can be easily verified that

   $$\phi_i(u) = \begin{cases} 
   \phi_i(w_{A(F)}) & \text{if } i \in A(F), \\
   0 & \text{if } i \in F \setminus A(F).
   \end{cases}$$
By (7), \((w - u)(D) = 0\) for every \(D \subsetneq F\). Therefore,
\[
\phi_i(w|_F - u) = \frac{w(F) - u(F)}{|F|} \quad \text{for every } i \in F.
\]
Thus
\[
\phi_i(w|_F) = \phi_i(u) + \frac{w(F) - u(F)}{|F|},
\]
or, equivalently,
\[
\phi_i(w|_F) = \begin{cases} 
\phi_i(w|_{A(F)}) + \frac{w(F) - u(F)}{|F|} & \text{if } i \in A(F), \\
\frac{w(F) - u(F)}{|F|} & \text{if } i \in F \setminus A(F), 
\end{cases}
\]
where (8) has been used. Taking into consideration that \(\lambda^F \ast x^F = \phi(w|_F)\) and \(\lambda^{A(F)} \ast x^{A(F)} = \phi(w|_{A(F)})\) it follows that
\[
x_i^F = \begin{cases} 
x_i^{A(F)} + \frac{w(F) - w(A(F))}{|F|\lambda_i} & \text{if } i \in A(F), \\
\frac{w(F) - w(A(F))}{|F|\lambda_i} & \text{if } i \in F \setminus A(F).
\end{cases}
\] (9)

Let \(y, z \in 0^F\) defined as
\[
y_i = \begin{cases} 
x_i^{A(F)} & \text{if } i \in A(F), \\
0 & \text{if } i \in F \setminus A(F),
\end{cases}
\]
\[
z_i = \frac{w(F) - w(A(F))}{|F|\lambda_i},
\]
for every \(i \in F\). Taking into account that \(x^{A(F)} \in \partial(V^A(A(F))) = \partial(V(A(F)))\), we have \(y \in \partial(V(A(F))) \times [0, 1]^{N \setminus \{0\}} \subseteq \partial(V(A(F)) \times (-\infty, 0]^{N \setminus \{0\}}) = \partial(V^A(A))\). Therefore, \(x^F = y + z\) where \(x^F, y \in \partial(V^A(A(F)))\) and \(z \in [0^+, 0^-] \cup \{0\}\). With these conditions, it is easy to verify that \(z = 0\). Hence, \(w(F) - w(A(F)) = 0\). Substituting in (9), we conclude that
\[
x_i^F = \begin{cases} 
x_i^{A(F)} & \text{if } i \in A(F), \\
0 & \text{if } i \in F \setminus A(F).
\end{cases}
\]

\[\square\]

**Theorem 4.6:** A configuration correspondence for NTU games with interior structure is equal to the Harsanyi configuration correspondence for NTU games with interior structure if and only if it satisfies the properties of efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives, consistency with the Shapley interior value and zero inessential games.

**Proof:** It will first be proved that the Harsanyi configuration correspondence for NTU games with interior structure satisfies the properties in the theorem.

**Efficiency.** Let \(V \in \Omega^N\), let \(\partial\) be an interior operator on \(N\), let \((x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(V, A)\) and \(E \in \text{aut}(A)\). Since \(\hat{\mathcal{H}}(V, A) = \mathcal{H}(V^A)\) and \(H\) satisfies efficiency it follows that \(x^E \in \partial(V^A(E)) = \partial(V(E))\).

**Conditional additivity.** Let \(V, W \in \Omega^N\) be such that \(V + W \in \Omega^N\) and let \(\partial\) be an interior operator on \(N\). Let \((x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(V, A) + \mathcal{H}(W, A)\) be such that \(x^E \in \partial((V + W)(E))\) for all \(E \in \text{aut}(A)\).
Let \((y^E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(V, A)\) and \((z^E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(W, A)\) be such that \(x^E = y^E + z^E\) for every \(E \in 2^N \setminus \{\emptyset\}\). Take \(F \in 2^N \setminus \{\emptyset\}\). In order to show that \(x^F \in \partial ((V^A + W^A)(F))\), we distinguish two cases:

(a) If \(A(F) = \emptyset\) it is known, from Proposition 4.5, that \(y^F = z^F = 0\) and hence \(x^F = 0\). Notice that \((V^A + W^A)(F) = (-\infty, 0]^F\). Therefore, \(x^F \in \partial ((V^A + W^A)(F))\).

(b) If \(A(F) \neq \emptyset\) it is known, from Proposition 4.5, that

\[
y^F_i = \begin{cases} y_i^{A(F)} & \text{if } i \in A(F), \\ 0 & \text{if } i \in F \setminus A(F), \end{cases}
\]

and

\[
z^F_i = \begin{cases} z_i^{A(F)} & \text{if } i \in A(F), \\ 0 & \text{if } i \in F \setminus A(F), \end{cases}
\]

whence it follows that

\[
x^F_i = \begin{cases} x_i^{A(F)} & \text{if } i \in A(F), \\ 0 & \text{if } i \in F \setminus A(F). \end{cases}
\]

By hypothesis, it is known that

\[
x^{A(F)} \in \partial \left( (V + W)(A(F)) \right).
\]

Moreover, it can be easily verified that \(V^A + W^A = (V + W)^A\). Therefore,

\[(V^A + W^A)(F) = (V + W)^A(F) = (V + W)(A(F)) \times (-\infty, 0]^F \setminus A(F).
\]

From (10), (11) and (12) it follows that \(x^F \in \partial ((V^A + W^A)(F))\).

We have \((x^E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V^A) + H(W^A)\) and \(x^E \in \partial ((V^A + W^A)(E))\) for every \(E \in 2^N \setminus \{\emptyset\}\). Since \(H\) satisfies conditional additivity, we conclude that

\[
(x^E)_{E \in 2^N \setminus \{\emptyset\}} \in H (V^A + W^A) = H ((V + W)^A) = \mathcal{H}(V + W, A).
\]

Scale covariance. Let \(V \in \Omega^N\), let \(A\) be an interior operator on \(N\) and let \(\alpha \in 0^N_++\). We have \(\mathcal{H}(\alpha * V, A) = H ((\alpha * V)^A)\). Moreover, it can be easily verified that \((\alpha * V)^A = \alpha * V^A\). Therefore, \(\mathcal{H}(\alpha * V, A) = H (\alpha * V^A)\) which, using that the Harsanyi configuration correspondence for NTU games satisfies scale covariance, is equal to

\[
\left\{ (\alpha^E * x^E)_{E \in 2^N \setminus \{\emptyset\}} : (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in H(V^A) \right\},
\]

or equivalently,

\[
\left\{ (\alpha^E * x^E)_{E \in 2^N \setminus \{\emptyset\}} : (x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(V, A) \right\}.
\]

Independence of irrelevant alternatives. Let \(V, W \in \Omega^N\), let \(A\) be an interior operator on \(N\) such that \(V(E) \subseteq W(E)\) for all \(E \in \text{aut}(A)\), and let \((x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \mathcal{H}(W, A)\) be such that \(x^E \in V(E)\) for all \(E \in \text{aut}(A)\). From \(V(E) \subseteq W(E)\) for all \(E \in \text{aut}(A)\), it easily derives that

\[
V^A(E) \subseteq W^A(E) \quad \text{for every } E \in 2^N \setminus \{\emptyset\}.
\]

Let \(F \in 2^N \setminus \{\emptyset\}\). In order to show that \(x^F \in V^A(F)\), we distinguish two cases:
(a) If \( A(F) = \emptyset \) it is known, from Proposition 4.5, that \( x^F = 0 \). In this case, \( V^A(F) = (-\infty, 0]^F \), hence \( x^F \in V^A(F) \).

(b) If \( A(F) \neq \emptyset \) it is known, from Proposition 4.5, that

\[
x^F = \begin{cases} x_i^{A(F)} & \text{if } i \in A(F), \\ 0 & \text{if } i \in F \setminus A(F). \end{cases}
\] (14)

By hypothesis, it is known that

\[ x^{A(F)} \in V(A(F)). \] (15)

From (14), (15) and \( V^A(F) = V(A(F)) \times (-\infty, 0]^F \setminus A(F) \) it follows that \( x^F \in V^A(F) \).

We have \( x^F \in V^A(E) \) for every \( E \in 2^N \setminus \emptyset \). From this statement, (13), \( (x^E)_{E \in 2^N \setminus \emptyset} \in H(W^A) \) and the fact that \( H \) satisfies independence of irrelevant alternatives it follows that \( (x^E)_{E \in 2^N \setminus \emptyset} \in H(V^A) = \mathcal{H}(V, A) \).

**Consistency with the Shapley interior value.** Let \( v \in G^N \) and let \( A \) be an interior operator on \( N \).

It is easy to verify that if \( V \in \Omega^N \) and \( \partial(V(N)) \) is a hyperplane, then \( |H(V)| = 1 \). We know that \( V^A(N) = V_v(N) \), which is a hyperplane. We obtain that \( |\mathcal{H}(V_v, A)| = |H(V_v^A)| = 1 \). Let \( \mathcal{H}(V_v, A) = \{(x^E)_{E \in 2^N \setminus \emptyset}\} \). Taking into account Proposition 4.5 (b), it suffices to prove that \( x^F = \Phi(v_{|F}, A_{|F}) \) for every \( F \in \text{aut}(A) \). Firstly, it is clear that \( \{(x^E)_{E \in 2^N \setminus \emptyset}\} \) is associated with the comparison vector \( \lambda = 1_N \). Let \( w \) be the TU game given by \( w(E) = \sum_i x_i^F \) for every \( E \in 2^N \setminus \emptyset \). We have that \( x^F \in \partial(V_v(F)) \) for every \( F \in \text{aut}(A) \). Therefore,

\[
w(F) = \sum_{i \in E} x_i^F = v(F) \quad \text{for every } F \in \text{aut}(A). \] (16)

By Proposition 4.5 (b), it is clear that

\[ w(E) = w(A(E)) \quad \text{for every } E \in 2^N \setminus \emptyset. \] (17)

From (16) to (17) we obtain that \( w = v^A \). For every \( F \in \text{aut}(A) \) we have that \( x^F = \phi(w_{|F}) = \Phi(v_{|F}, A_{|F}) \).

**Zero inessential games.** Let \( V \in \Omega^N \) and let \( A \) be an interior operator on \( N \) such that \( 0 \in \partial(V(F)) \) for all \( F \in \text{aut}(A) \). It is clear that \( 0 \in \partial(V^A(E)) \) for every \( E \in 2^N \setminus \emptyset \). Since the Harsanyi configuration correspondence for NTU games satisfies the property of zero inessential games it follows that

\[
(0)_{E \in 2^N \setminus \emptyset} \in H(V^A) = \mathcal{H}(V, A).
\]

We have proved that \( \mathcal{H} \) satisfies the properties in the theorem. Now, it will be proved that these properties uniquely determine the Harsanyi configuration correspondence for NTU games with interior structure.

Let \( \Upsilon \) and \( \Psi \) be configuration correspondences for NTU games with interior structure satisfying the properties of efficiency, conditional additivity, scale covariance, independence of irrelevant alternatives, consistency with the Shapley interior value and zero inessential games. It must be proved that \( \Upsilon = \Psi \). Take \( V \in \Omega^N \) and \( A \) an interior operator on \( N \). We want to verify that \( \Upsilon(V, A) = \Psi(V, A) \). By symmetry, it suffices to prove one inclusion.

Let \( (x^E)_{E \in 2^N \setminus \emptyset} \in \Upsilon(V, A) \). Consider \( W_1 \in \Omega^N \) given by

\[ W_1(E) = V(E) - \{x^E\} \quad \text{for every } E \in 2^N \setminus \emptyset. \]
Since $\Upsilon$ satisfies efficiency we have $x^F \in \partial(V(F))$ for every $F \in \text{aut}(A)$, whence we obtain that $0 \in \partial(W_1(F))$ for every $F \in \text{aut}(A)$. From the property of zero inessential games, it is derived that

$$(0)_{E \in 2^N \setminus \emptyset} \in \Psi(W_1, A).$$

(18)

From properties (i), (ii) and (iii) of the games in $\Omega^N$, it follows that there exists $\lambda \in 0^N_{++}$ such that the supporting hyperplane of $V(N)$ at $x^N$ is $\{y \in 0^N : \lambda \cdot y = \lambda \cdot x^N\}$. We have

$$\lambda \cdot y \leq \lambda \cdot x^N \quad \text{for all } y \in V(N).$$

Take $W_2 \in \Omega^N$ defined by

$$W_2(E) = \begin{cases} V(N) & \text{if } E = N, \\ \{y \in 0^E : y \leq x^E\} & \text{if } E \subsetneq N, E \neq \emptyset. \end{cases}$$

Since $\Upsilon$ satisfies independence of irrelevant alternatives, we have $(x^E)_{E \in 2^N \setminus \emptyset} \in \Upsilon(W_2, A)$. Using scale covariance, it follows that

$$(\lambda^E \cdot x^E)_{E \in 2^N \setminus \emptyset} \in \Upsilon(\lambda \ast W_2, A).$$

(19)

Let $V_0$ be the NTU game corresponding to the TU game that is identically zero. From the fact that $\Upsilon$ satisfies consistency with the Shapley interior value, it follows that

$$\Upsilon(V_0, A) = \{(0)_{E \in 2^N \setminus \emptyset}\}.$$  

(20)

From (19) and (20), we obtain that $(\lambda^E \cdot x^E)_{E \in 2^N \setminus \emptyset} \in \Upsilon(\lambda \ast W_2, A) + \Upsilon(V_0, A)$. Notice that $\lambda \ast W_2 + V_0 = V_r$ where $v$ is the TU game given by $v(F) = \lambda^F \cdot x^F$ for every $F \in 2^N \setminus \emptyset$. From conditional additivity, it follows that $(\lambda^E \cdot x^E)_{E \in 2^N \setminus \emptyset} \in \Upsilon(V_r, A)$. Since $\Upsilon$ and $\Psi$ satisfy the property of consistency with the Shapley interior value we have $\Upsilon(\lambda \ast W_2 + V_0, A) = \Psi(\lambda \ast W_2 + V_0, A)$. Therefore,

$$(\lambda^E \cdot x^E)_{E \in 2^N \setminus \emptyset} \in \Psi(\lambda \ast W_2 + V_0, A).$$

(21)

Consider $W_3 \in \Omega^N$ given by

$$W_3(E) = \begin{cases} \{y \in 0^N : \lambda \cdot y \leq \lambda \cdot x^N\} & \text{if } E = N, \\ \{y \in 0^E : y \leq x^E\} & \text{if } E \subsetneq N, E \neq \emptyset. \end{cases}$$

It is clear that

$$(\lambda \ast W_3)(E) \subseteq (\lambda \ast W_2 + V_0)(E) \quad \text{for all } E \in 2^N \setminus \emptyset.$$  

(22)

From (21), (22), the fact that $\lambda^E \cdot x^E \in (\lambda \ast W_3)(E)$ for all $E \in 2^N \setminus \emptyset$ and the property of independence of irrelevant alternatives it follows that

$$(\lambda^E \cdot x^E)_{E \in 2^N \setminus \emptyset} \in \Psi(\lambda \ast W_3, A),$$

which, from scale covariance, implies that

$$(x^E)_{E \in 2^N \setminus \emptyset} \in \Psi(W_3, A).$$

(23)

From (18), (23) and the property of conditional additivity, we conclude that

$$(x^E)_{E \in 2^N \setminus \emptyset} \in \Psi(W_1 + W_3, A).$$
But notice that
\[(W_1 + W_3)(E) = \begin{cases} \{y \in \mathbb{R}^N : \lambda \cdot y \leq \lambda \cdot x^N\} & \text{if } E = N, \\ V(E) & \text{if } E \subsetneq N, E \neq \emptyset. \end{cases}\]

Hence, \(V(E) \subseteq (W_1 + W_3)(E)\) for every \(E \in 2^N \setminus \{\emptyset\}\). Finally, using the property of independence of irrelevant alternatives we conclude that \((x^E)_{E \in 2^N \setminus \{\emptyset\}} \in \Psi(V, A)\).

In practice, given \(V \in \Omega^N\) and \(A\) an interior operator on \(N\), we do not need to obtain \(V^A\) to calculate \(\mathcal{H}(V, A)\). From Remark 1 and Proposition 4.5, we can derive an alternative definition of \(\mathcal{H}(V, A)\) that does not involve the restricted game \(V^A\). We give that definition in the following remark.

**Remark 2**: Let \(V \in \Omega^N\) and \(A\) an interior operator on \(N\). A pay-off configuration \((x^E)_{E \in 2^N \setminus \{\emptyset\}}\) belongs to \(\mathcal{H}(V, A)\) if there exists \(\lambda \in \mathbb{R}^N_{++}\) such that

1. \(x^E \in \partial(V(E))\) for all \(E \in aut(A)\),
2. \(\lambda \cdot x^N = \max \{\lambda \cdot y : y \in V(N)\}\),
3. if \(w\) is the TU game given by \(w(F) = \lambda^F \cdot x^F\) for every \(F \in 2^N \setminus \{\emptyset\}\), then \(\lambda^E \cdot x^E = \phi(w|E)\) for every \(E \in aut(A)\),
4. \(x^E = 0\) for all \(E \in 2^N \setminus \{\emptyset\}\) with \(A(E) = \emptyset\),
5. For every \(E \in 2^N\) with \(A(E) \neq \emptyset\),

\[x^E_i = \begin{cases} x^A_i & \text{if } i \in A(E), \\ 0 & \text{if } i \in E \setminus A(E). \end{cases}\]

**Example 4.7**: Let \(N = \{1, 2, 3\}\). Let \(V\) be the NTU game on \(N\) defined as

\[
\begin{align*}
V\left(\{i\}\right) &= \left\{ z_i \in \mathbb{R}^{\{i\}} : z_i \leq 1 \right\} \quad \text{for every } i \in N, \\
V\left(\{i,j\}\right) &= \left\{ z \in \mathbb{R}^{\{i,j\}} : z \leq (2,2) \right\} \quad \text{for every } i,j \in N \text{ with } i \neq j, \\
V\left(N\right) &= \left\{ (z_1, z_2, z_3) \in \mathbb{R}^N : z_1 + z_2 + z_3 \leq 6 \right\}. 
\end{align*}
\]

Let \(A\) be the interior operator on \(N\) defined as

\[A(E) = \begin{cases} E \setminus \{2,3\} & \text{if } 1 \notin E, \\ E & \text{otherwise}. \end{cases}\]

Notice that \((N, A)\) is a hierarchical structure in which player 1 has veto power over players 2 and 3. It can be represented by the following digraph

```
1

2

3
```

Let us calculate \(\mathcal{H}(V, A)\). Since \(\partial(V(N))\) is a hyperplane, it is plain to see that \(\mathcal{H}(V, A)\) contains exactly one pay-off configuration \((x^E)_{E \in 2^N \setminus \{\emptyset\}}\). From the expression of \(V(N)\), we can derive that
\((x^E)_{E\in 2^N\setminus\{\emptyset\}}\) is associated with the comparison vector \(\lambda = (1, 1, 1)\). Let \(w\) be the TU game given by 
\(w(E) = \lambda^E \cdot x^E\) for every nonempty \(E \subseteq N\). Let us calculate \((x^E)_{E\in 2^N\setminus\{\emptyset\}}\) using Remark 2.

From Remark 2 (1), it follows that
\[x^{(1)} = 1,\]
and from Remark 2 (4), it follows that
\[
\begin{align*}
x^{(2)} &= 0, \\
x^{(3)} &= 0, \\
x^{(2,3)} &= (0, 0).
\end{align*}
\]

Then
\[
\begin{align*}
w(\{1\}) &= \lambda^{(1)} \cdot x^{(1)} = 1, \\
w(\{2\}) &= \lambda^{(2)} \cdot x^{(2)} = 0, \\
w(\{3\}) &= \lambda^{(3)} \cdot x^{(3)} = 0, \\
w(\{2, 3\}) &= \lambda^{(2,3)} \cdot x^{(2,3)} = 0.
\end{align*}
\]

We have
\[
\phi(w_{\{1,2\}}) = \left(1 + \frac{w(\{1, 2\}) - 1}{2}, \frac{w(\{1, 2\}) - 1}{2}\right).
\]
Hence,
\[
x^{(1,2)} = \left(1 + \frac{w(\{1, 2\}) - 1}{2}, \frac{w(\{1, 2\}) - 1}{2}\right).
\]

From this fact and Remark 2 (1), we conclude that
\[
x^{(1,2)} \in \partial \left(V(\{1, 2\}) \right) \cap \left\{ (1 + \alpha, \alpha) \in \mathbb{R}^{(1,2)} : \alpha \in \mathbb{R} \right\} = \{ (2, 1) \}.
\]

Therefore,
\[
x^{(1,2)} = (2, 1),
\]
and, similarly,
\[
x^{(1,3)} = (2, 1).
\]

Thus,
\[
\begin{align*}
w(\{1, 2\}) &= \lambda^{(1,2)} \cdot x^{(1,2)} = 3, \\
w(\{1, 3\}) &= \lambda^{(1,3)} \cdot x^{(1,3)} = 3.
\end{align*}
\]

Moreover, from \(x^N \in \partial(V(N))\) we conclude that \(\lambda \cdot x^N = 6\). Therefore,
\[
w(\{1, 2, 3\}) = 6.
\]

Finally,
\[
\phi(w) = \left(\frac{10}{3}, \frac{4}{3}, \frac{4}{3}\right).
\]
Hence,
\[
x^{(1,2,3)} = \left(\frac{10}{3}, \frac{4}{3}, \frac{4}{3}\right).\]
5. Conclusions

We have studied NTU games with restricted cooperation. In order to model the limitations on cooperation, we have used the concept of interior operator, which had been used in the literature on TU games with restricted cooperation. Given an NTU game and an interior operator, an NTU restricted game is defined. Then, we have applied one of the best-known NTU values, the Harsanyi value, to the restricted game, obtaining in this way a value for NTU games with restricted cooperation. By achieving a characterization of this value, we have proved that it satisfies some desirable properties. Taking into account that NTU games extend different kinds of games, such as TU games, market games or pure bargaining games, and that interior operators extend other combinatorial structures used in the literature to model restrictions on cooperation, such as antimatroids or permission structures, the value proposed can be used to obtain pay-off vectors in a variety of cooperative situations. Perhaps the most interesting contribution of the paper is that it has been shown that the idea of defining a restricted game that gathers the information from both the original game and the combinatorial structure and then applying a well-known value, which had been extensively used in the study of TU games with restricted cooperation, can be adapted to the case of NTU games. The application of this idea to other NTU values and other combinatorial structures could lead to other values for NTU games with restricted cooperation.

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