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Theory and Methodology

Axioms for the Shapley value on convex geometries

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Abstract

The purpose of this article is an extension of Shapley's value for games with restricted cooperation. The classical model of cooperative game where every subset of players is a feasible coalition may be unrealistic. The feasible coalitions in our model will be the *convex sets*, i.e., those subsets of players belonging to a convex geometry $\mathscr{L}$. In the last section, we apply this model to several examples about the power in the Council of Ministers of the European Union.

Keywords: Shapley value; Convex geometry

1. Introduction

The Shapley value [1] has been generalized by Myerson [2], in the context of the restricted games by *communication situations*, defined by a game (N, v) and the family of all subsets of N that induced a connected subgraph of the graph $G = (N,E)$. There is another extension by Faigle and Kern [3], the "cooperative games under precedence constraints". In the model of Faigle and Kern the games are defined on distributive lattices of subsets of players, $v : \mathscr{D} \longrightarrow \mathbb{R}$, but there may exist some player $i \in N$ such that the coalition $\{i\} \notin \mathscr{D}$. Then, the restricted games are not included in this model.

Note that every distributive lattice is a convex geometry [4], and we can suppose that every individual player $\{i\} \in \mathscr{L}$. Furthermore, the restricted games by connected block graphs are games on convex geometries.

First, we define the concept of *convex geometry* and describe its fundamental properties. In Sections 2 and 3, we give four axioms for an allocation rule and show that there is a unique "Shapley value" that satisfies the axioms. Finally, we apply this model to analyze the power of the countries in the European Union.

Let N be a finite set of cardinality n and consider a family $\mathscr{L}$ of subsets of N with the properties:
(1) $\emptyset \in \mathscr{L}$ and $N \in \mathscr{L}$,
(2) $A \in \mathscr{L}$ and $B \in \mathscr{L}$ imply that $A \cap B \in \mathscr{L}$.

The family $\mathscr{L}$ gives rise to the operator $- : 2^N \longrightarrow 2^N$ defined by

$$A \mapsto \overline{A} := \bigcap \{C \in \mathscr{L} \mid A \subseteq C\}.$$

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It is easy to check that the operator $-$ has the properties of a closure operator: $A \subseteq \bar{A}$, $\bar{\bar{A}} = \bar{A}$, and $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$, for all $A, B \subseteq N$, and the additional condition that $\emptyset = \overline{\emptyset}$. Conversely, every closure operator with the above conditions defines a family $\mathscr{L} \subseteq 2^N$ with properties (1) and (2) as the family of its closed sets

$$\mathscr{L} = \mathscr{L}(N, -) := \{A \subseteq N \mid A = \overline{A}\}.$$

If $(N, -)$ is a closure space then $\mathscr{L} \subseteq 2^N$, ordered by inclusion, is a complete lattice in which meet and join operations are defined by

$$\forall A,\ B \in \mathscr{L}: \quad A \wedge B = A \cap B,\ A \vee B = \overline{A \cup B}.$$

Definition 1. $\mathscr{L} \subseteq 2^N$ is a convex geometry if it satisfies the extension property: For every $S \in \mathscr{L}$ and $S \neq N$, there exists $i \in N \setminus S$ such that $S \cup i \in \mathscr{L}$.

We call the closed sets in a convex geometry *convex sets*. For a convex set $S \in \mathscr{L}$ an element $i \in S$ is an extreme point of S if $S \setminus i \in \mathscr{L}$. In a convex geometry $ex(S) \neq \emptyset$ and the closure $\overline{ex(S)} = S$ (see [4]).

A *maximal chain* of the convex geometry $\mathscr{L} \subseteq 2^N$ is an ordered collection of convex sets that is not contained in any larger chain. From the definition and by induction, Edelman and Jamison [4] showed that every maximal chain contains $n+1$ convex sets

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{n-1} \subset S_n = N,$$

and the cardinal $|S_k| = k$, for all $k = 0, 1, \ldots, n$.

Example 1. Let $N = \{1,2,3,4\}$ be a set and we consider

$$\mathscr{L} = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, N\}$$
and
$$\mathscr{F} = \{\emptyset, \{1\}, \{2\}, \{1,3\}, \{2,4\}, N\}.$$

The collection $\mathscr{L}$ is a convex geometry such that

$$\overline{\{1\}} = \{1\}, \quad \overline{\{2\}} = \{1,2\},$$
$$\overline{\{3\}} = \{1,2,3\}, \quad \overline{\{4\}} = N,$$

and $ex(\{1,\ldots,k\}) = k$, $1 \leqslant k \leqslant 4$. For the collection $\mathscr{F}$ fails the extension property and $ex(N) = \emptyset$.

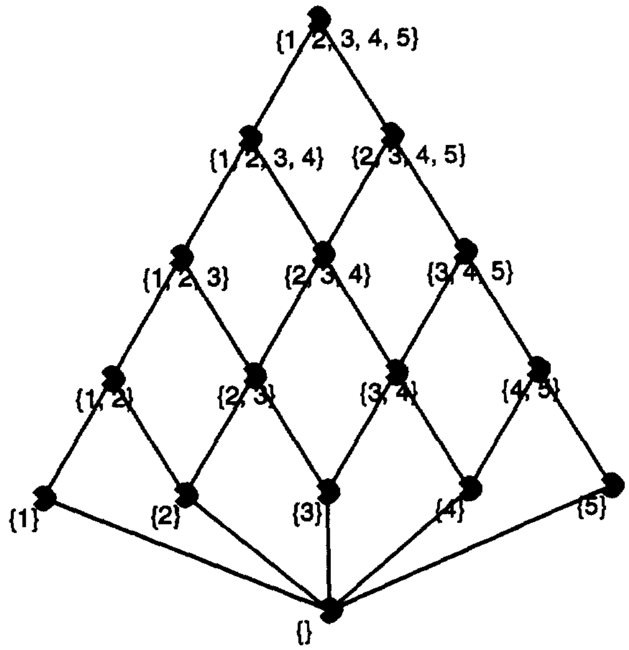
Example 2. A graph $G = (N, E)$ is connected if any two vertices can be joined by a path. A maximal connected subgraph of G is a *component* of G. A *cutvertex* is one whose removal increases the number of components, and a *bridge* is an edge with the same property. A graph is *2-connected* if it is connected, has at least 3 vertices and contains no cutvertex. A subgraph B of a graph G is a *block* of G if either B is a bridge or else it is a maximal 2-connected subgraph of G. A graph G is a *block graph* if every block is a complete graph.

A communication situation is a triple (N, G, v), where (N, v) is a game and $G = (N, E)$ is a graph. This concept was first introduced in [2], and investigated in [5]. If $G = (N, E)$ is a connected block graph and $\mathscr{L} = \{S \subseteq N \mid (S, E(S))$ is a connected subgraph of G$\}$, then we have [4, Theorem 3.7] that $\mathscr{L}$ is a convex geometry.

Example 3. Let $(P, \leqslant)$ be a partially ordered set, i.e., a set with a binary relation '$\leqslant$' which satisfies the reflexive, antisymmetry and transitivity properties. A subset S of $(P, \leqslant)$ is convex whenever $a \in S$, $b \in S$ and $a \leqslant b$ imply that the interval $[a,b] = \{x \in P \mid a \leqslant x \leqslant b\} \subseteq S$. The convex subsets of P form a closure system $Co(P)$. If P (or, equivalently $Co(P)$) is finite, then each element is between a maximal and a minimal one. If $C \in Co(P)$ then $ex(C)$ is the union of the maximal and minimal elements of C. Moreover, $Co(P)$ is a convex geometry ([6, Theorem 3]). For instance, if we consider the linear order defined by $(P, \leqslant) = \{1 < 2 < 3 < 4 < 5\}$ then the Hasse diagram of the lattice

$$\begin{aligned} Co(P) = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \\ &\{2,3\}, \{3,4\}, \{4,5\}, \{1,2,3\}, \\ &\{2,3,4\}, \{3,4,5\}, \{1,2,3,4\}, \\ &\{2,3,4,5\}, \{1,2,3,4,5\}\} \end{aligned}$$

is the following.



A *cooperative game* is a function $v : 2^N \longrightarrow \mathbb{R}$, with $v(\emptyset) = 0$. The *players* are the elements of N and the *coalitions* are the subsets $S \subseteq N$.

Definition 2. A game on a convex geometry $\mathscr{L}$ is a function

$$v : \mathscr{L} \longrightarrow \mathbb{R}, \quad v(\emptyset) = 0.$$

The coalitions are the convex sets of $\mathscr{L}$ and the players are the elements $i \in N$. Let $\Gamma(\mathscr{L})$ be the vector space over $\mathbb{R}$ of all games on the convex geometry $\mathscr{L} \subseteq 2^N$. A game on a convex geometry is called simple, monotonic, superadditive or convex when $v : \mathscr{L} \longrightarrow \mathbb{R}$ satisfies the corresponding property for the partial order, and the join and meet operations.

Example 4. Let $(P, \leqslant)$ be a partially ordered set. For any $X \subseteq P$,

$$X \mapsto \overline{X} := \{y \in P \mid y \leqslant x \quad \text{for some } x \in X\}$$

defines a closure operator on P. Its closed sets are the *order ideals* (down sets) of P, and we denote this lattice $J(P)$. Since the union and intersection of order ideals is again an order ideal, it follows that $J(P)$ is a sublattice of 2^P. Then $J(P)$ is a distributive lattice and so, $J(P)$ is a convex geometry closed under set-union and $ex(S)$ is the set of all maximal points $\max(S)$ of the subposet $S \in J(P)$. Then the game $(\mathscr{C}, v)$ of Faigle and Kern [3], where $\mathscr{C}$ is the family of down sets of P is a game on a distributive lattice.

2. Axioms for values on convex geometries

We will follow the strategy of Weber [7] for the axiomatic approach to the Shapley value for games on convex geometries. First, we consider the following simple games on $\mathscr{L}$. For any $T \in \mathscr{L}$, $T \neq \emptyset$ the *upper game*, denoted $\zeta_T : \mathscr{L} \longrightarrow \{0, 1\}$, is defined by

$$\zeta_T(S) := \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

The *identity game* δ_T is defined by

$$\delta_T(S) := \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{if } S \neq T. \end{cases}$$

The collections of these games are two different basis of the vector space $\Gamma(\mathscr{L})$. Faigle and Kern [3] observed that

$$\zeta_T = \sum_{\{S \in \mathscr{L} \mid S \supseteq T\}} \delta_S.$$

Let Φ be a map

$$\Phi : \Gamma(\mathscr{L}) \longrightarrow \mathbb{R}^n : v \mapsto (\Phi_1(v), \ldots, \Phi_n(v)).$$

First, we consider the linearity property for the *values* $\Phi_i(v)$ of every player $i \in N$.

Linearity axiom: For all $\alpha, \beta \in \mathbb{R}$, and $v, w \in \Gamma(\mathscr{L})$ we have

$$\Phi_i(\alpha v + \beta w) = \alpha \Phi_i(v) + \beta \Phi_i(w) \quad \text{for every } i \in N.$$

Theorem 1. *Let $\Phi_i : \Gamma(\mathscr{L}) \longrightarrow \mathbb{R}$ be a value for i which satisfies the linearity axiom. Then there exists an unique set of coefficients $\{a^i_S \mid S \in \mathscr{L}, S \neq \emptyset\}$ such that*

$$\Phi_i(v) = \sum_{S \in \mathscr{L}} a^i_S v(S) \quad \textit{for all } v \in \Gamma(\mathscr{L}).$$

Proof. The collection $\{\delta_S \mid S \in \mathscr{L},\ S \neq \emptyset\}$ is a basis of the vector space $\Gamma(\mathscr{L})$. We have for every game v, that $v = \sum_{S\in\mathscr{L}} v(S)\delta_S$. Let $a^i_S = \Phi_i(\delta_S)$ for all $S \in \mathscr{L}$. Then the linearity axiom implies

$$\Phi_i(v) = \sum_{S\in\mathscr{L}} \Phi_i(\delta_S)v(S) = \sum_{S\in\mathscr{L}} a^i_S v(S). \qquad \square$$

We define the concept of *dummy player.*

Definition 3. The player $i \in N$ is a dummy in the game $v \in \Gamma(\mathscr{L})$ if for every convex $S \in \mathscr{L}$ such that $i \in ex(S)$ we have

$$v(S) - v(S \setminus i) = \begin{cases} v(i) & \text{if } \{i\} \in \mathscr{L}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1. A different definition has been suggested by Faigle and Kern [3].

We need some properties of the dummy players in the upper and identity games.

Proposition 1. *Let $\mathscr{L}$ be a convex geometry and let $S \in \mathscr{L}$ be a convex set. Then:*
1. *If $i \notin ex(S)$ then i is a dummy player in the upper game ζ_S.*
2. *If $i \in S \setminus ex(S)$ then i is a dummy player in the identity game δ_S.*

Proof. Let $i \in N$ be such that $i \notin ex(S)$. If $\{i\} \in \mathscr{L}$ then $S \neq \{i\}$, hence $\zeta_S(\{i\}) = 0$ and $\delta_S(\{i\}) = 0$.

(1) We suppose that the player $i \notin ex(S)$ is not a dummy in ζ_S. Then there exists $T \in \mathscr{L}$, and $i \notin T$ such that $T \cup i \in \mathscr{L}$ and satisfies $\zeta_S(T \cup i) \neq \zeta_S(T)$. Then $\zeta_S(T \cup i) = 1$ and $\zeta_S(T) = 0$, hence $S \subseteq T \cup i$ and $S \nsubseteq T$. Thus $S \setminus i = S \cap T \in \mathscr{L}$, hence we obtain $i \in ex(S)$, which is a contradiction.

(2) If $T \in \mathscr{L}$, $i \notin T$ and $T \cup i \in \mathscr{L}$ such that $\delta_S(T \cup i) \neq \delta_S(T)$ then $\delta_S(T \cup i) = 1$ or $\delta_S(T) = 1$. If $S = T \cup i$ then $S \setminus i = T \in \mathscr{L}$. If $S = T$ then $i \in T$, which contradicts the hypothesis. $\square$

Dummy axiom: If the player $i \in N$ is a dummy in $v \in \Gamma(\mathscr{L})$, then

$$\Phi_i(v) = \begin{cases} v(i) & \text{if } \{i\} \in \mathscr{L}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2. *Let $\Phi_i : \Gamma(\mathscr{L}) \longrightarrow \mathbb{R}$ be a value for i defined by $\Phi_i(v) = \sum_{S\in\mathscr{L}} a^i_S v(S)$ which satisfies the dummy axiom. Then for every game v,*

$$\Phi_i(v) = \sum_{\{S\in\mathscr{L} \mid i\in ex(S)\}} a^i_S[v(S) - v(S \setminus i)].$$

Moreover, if $\{i\} \in \mathscr{L}$ then $\sum_{\{S\in\mathscr{L} \mid i\in ex(S)\}} a^i_S = 1$.

Proof. Let $E_i : \Gamma(\mathscr{L}) \longrightarrow \mathbb{R}$, be defined for $i \in N$ by

$$E_i(v) := \sum_{\{S\in\mathscr{L} \mid i\in ex(S)\}} a^i_S[v(S) - v(S \setminus i)].$$

The operators E_i and Φ_i are linear and the upper games form a basis of $\Gamma(\mathscr{L})$. Then it will enough be to show that $\Phi_i(\zeta_T) = E_i(\zeta_T)$, for every convex set $T \in \mathscr{L}$, $T \neq \emptyset$. Fix $i \in N$ and $T \in \mathscr{L}$, and consider two cases.

First, if $i \notin ex(T)$ then Proposition 1 implies that i is a dummy player in the game ζ_T. Then $\zeta_T(S) - \zeta_T(S \setminus i) = 0$ for all $S \in \mathscr{L}$, such that $i \in ex(S)$ because $\zeta_T(\{i\}) = 0$, when $\{i\} \in \mathscr{L}$. Therefore, the definition of E_i yields $E_i(\zeta_T) = 0$. Furthermore, the dummy axiom implies that $\Phi_i(\zeta_T) = 0$.

Now, if we suppose that $i \in ex(T)$ then $i \in T$, hence $\zeta_T(S \setminus i) = 0$ and we obtain the equivalence

$\zeta_T(S) - \zeta_T(S \setminus i) = 1$ if and only if
$S \in \mathscr{L}$ and $S \supseteq T$.

Observe that

$$\begin{aligned} E_i(\zeta_T) &= \sum_{\{S\in\mathscr{L} \mid i\in ex(S),\ S\supseteq T\}} a^i_S \\ &= \sum_{\{S\in\mathscr{L} \mid i\in ex(S),\ S\supseteq T\}} \Phi_i(\delta_S) \\ &= \Phi_i\left(\sum_{\{S\in\mathscr{L} \mid S\supseteq T\}} \delta_S\right) \\ &= \Phi_i(\zeta_T), \end{aligned}$$

because $\Phi_i(\delta_S) = 0$, when $i \in S \setminus ex(S)$.

Finally, if $\{i\} \in \mathscr{L}$ then i is a dummy in the game $\zeta_{\{i\}}$. Let T be a convex set such that $i \notin T$ and $T \cup i \in \mathscr{L}$. Then we have that $\zeta_{\{i\}}(T \cup i) = 1$, $\zeta_{\{i\}}(T) = 0$, and $\zeta_{\{i\}}(\{i\}) = 1$.

Because Φ_i satisfies the dummy axiom, it follows from the above equation that

$$\sum_{\{S\in\mathscr{L} \mid i\in ex(S)\}} a_S^i = \Phi_i(\zeta_{\{i\}}) = \zeta_{\{i\}}(\{i\}) = 1. \qquad \square$$

We consider a convex geometry with $\{i\} \notin \mathscr{L}$ for some $i \in N$. In this case, note that

$$\sum_{\{S\in\mathscr{L} \mid i\in ex(S)\}} a_S^i = \sum_{\{S\in\mathscr{L} \mid i\in S\}} a_S^i = \Phi_i\left(\zeta_{\overline{\{i\}}}\right),$$

where the upper game on the closure of $\{i\}$ satisfies

$$\zeta_{\overline{\{i\}}}(T) = \begin{cases} 1 & \text{if } \overline{\{i\}} \subseteq T, \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} 1 & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain the following result by application of Theorems 1 and 2.

Theorem 3. *Let $\Phi_i : \Gamma(\mathscr{L}) \longrightarrow \mathbb{R}$ be a value for i which satisfies the linearity and dummy axioms. Then for every game v, there is an unique set of coefficients $\{a_S^i \mid S \in \mathscr{L},\ i \in ex(S)\}$ such that*

$$\Phi_i(v) = \sum_{\{S\in\mathscr{L} \mid i\in ex(S)\}} a_S^i[v(S) - v(S \setminus i)].$$

Moreover, if $\Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1$ then $\sum_{\{S\in\mathscr{L} \mid i\in ex(S)\}} a_S^i = 1$.

Remark 2. For convex geometries which satisfy $\{i\} \in \mathscr{L}$ for all $i \in N$, the condition $\Phi_i\left(\zeta_{\overline{\{i\}}}\right) = 1$ is not necessary, because it follows from dummy axiom.

If the vector of the values for the players $\Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v))$, is a distribution of the available resources to the coalition N, then Φ satisfies the following axiom:

Efficiency axiom: If N is the set of all players of $v \in \Gamma(\mathscr{L})$ then

$$\sum_{i\in N} \Phi_i(v) = v(N).$$

The efficiency axiom implies the following properties for the coefficients of the values that satisfy linearity and dummy axioms.

Theorem 4. *Let $\Phi : \Gamma(\mathscr{L}) \longrightarrow \mathbb{R}^n$ be a value defined for all game v and for all $i \in N$ by*

$$\Phi_i(v) = \sum_{\{S\in\mathscr{L} \mid i\in ex(S)\}} a_S^i[v(S) - v(S \setminus i)].$$

Then the value Φ satisfies the efficiency axiom if and only if

$$\sum_{i\in ex(N)} a_N^i = 1, \quad \text{and}$$

$$\sum_{i\in ex(S)} a_S^i = \sum_{\{j\notin S \mid S\cup j\in\mathscr{L}\}} a_{S\cup j}^j,$$

$$\forall S \in \mathscr{L},\ S \neq \emptyset,\ S \neq N.$$

Proof. For any $v \in \Gamma(\mathscr{L})$ we have

$$\sum_{i\in N} \Phi_i(v) = \sum_{i\in N} \sum_{\{S\in\mathscr{L} \mid i\in ex(S)\}} a_S^i[v(S) - v(S \setminus i)]$$

$$= \sum_{S\in\mathscr{L}} v(S) \left[\sum_{i\in ex(S)} a_S^i - \sum_{\{j\notin S \mid S\cup j\in\mathscr{L}\}} a_{S\cup j}^j \right].$$

If the coefficients satisfy the relations, then $\sum_{i\in N} \Phi_i(v) = v(N)$ hence Φ satisfies the efficiency axiom.

Conversely, fix a nonempty convex set $T \in \mathscr{L}$, and consider the identity game δ_T. If we apply the previous equation to the game δ_T, then we have

$$\sum_{i\in N} \Phi_i(\delta_T) = \begin{cases} \sum_{i\in ex(N)} a_N^i & \text{if } T = N, \\ \sum_{i\in ex(S)} a_S^i - \sum_{\{j\notin S \mid S\cup j\in\mathscr{L}\}} a_{S\cup j}^j & \text{if } T = S \neq N. \end{cases}$$

Thus, if Φ satisfies the efficiency axiom then the relations are true. $\square$

3. Axioms for the Shapley value

The classical characterization of the Shapley value is as the only value that satisfies the carrier, symmetry and additivity on the class of all super-additive games [1]. For the class of all games, Weber [7, Theorem 15] considered linearity, dummy, symmetry, and efficiency axioms, and proved the uniqueness of the Shapley value. If (N, v) is a game then the Shapley value for the player $i \in N$ is

$$\Phi_i(N, v) = \sum_{\{S \in 2^N \mid i \in S\}} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S \setminus i)],$$

where $n = |N|$ and $s = |S|$.

Edelman and Jamison [4] defined a *compatible ordering* of a convex geometry $\mathscr{L} \subseteq 2^N$ as a total ordering of the elements of N, $i_1 < i_2 < \cdots < i_n$ such that the set

$$\{i_1, i_2, \ldots, i_k\} \in \mathscr{L} \quad \text{for all } 1 \leqslant k \leqslant n.$$

A compatible ordering of $\mathscr{L}$ corresponds exactly to a maximal chain in $\mathscr{L}$. Denote by $c([T, S])$ the number of maximal chains from T to S, where $T \subset S$ and $c(S) := c([\emptyset, S])$ is the number of maximal chains from $\emptyset$ to $S \neq \emptyset$. Furthermore, $c([S, S]) = 1$ for all $S \in \mathscr{L}$.

Definition 4. Let $v \in \Gamma(\mathscr{L})$ be a game on a convex geometry. The Shapley value for the player $i \in N$ is given by

$$\Phi_i(v) := \sum_{\{S \in \mathscr{L} \mid i \in ex(S)\}} \frac{c(S \setminus i)\, c([S, N])}{c(N)}[v(S) - v(S \setminus i)].$$

We will denote the set of all compatible orderings by $\mathscr{C}(\mathscr{L})$. Given an element $i \in N$ and a compatible ordering C of $\mathscr{L}$, let

$$C(i) := \{j \in N \mid j \leqslant i \text{ in } C\}.$$

Let $S \in \mathscr{L}$ and $i \in S$. Faigle and Kern [3] defined the *hierarchical strength* $h_S(i)$ of i in S to be

$$h_S(i) := \frac{|\{C \in \mathscr{C}(\mathscr{L}) \mid C(i) \cap S = S\}|}{|\mathscr{C}(\mathscr{L})|},$$

i.e., $h_S(i)$ is the average number of compatible orderings of $\mathscr{L}$ in which i is the last member of S in the ordering. Note that $h_S(i) \neq 0 \Leftrightarrow i \in ex(S)$. They proposed the following axiom for the Shapley value on distributive lattices.

Hierarchical strength axiom: For any $S \in \mathscr{L}$ and $i, j \in S$,

$$h_S(i)\Phi_j(\zeta_S) = h_S(j)\Phi_i(\zeta_S).$$

Now, we introduce a new axiom, in which the value of the player depends of the position on lattice structure of the convex geometry.

Chain axiom: For any nonempty $S \in \mathscr{L}$ and $i, j \in ex(S)$,

$$c(S \setminus i)\Phi_j(\delta_S) = c(S \setminus j)\Phi_i(\delta_S).$$

We need this axiom because in the model of convex geometries the hypothesis of symmetry for the players is not applied. To interpret the chain axiom, assume that $\mathscr{L} = 2^N$. Then, the coefficients of the value in Theorem 2 satisfy $a_i^S = a_j^S$ for every pair $i, j \in S$ and for all $S \subseteq N$. The same property is consequence of the classical symmetry axiom (see [7, Theorem 10]).

By using our previous results, we prove the following characterization for the Shapley value on convex geometries.

Theorem 5. *The Shapley value is the unique function* $\Phi : \Gamma(\mathscr{L}) \longrightarrow \mathbb{R}^n$ *that satisfies the linearity, dummy, efficiency, and chain axioms.*

Proof. The Shapley value satisfies the axioms. Let Φ be a function that satisfies the linearity, dummy, efficiency, and chain axioms. It follows from Theorems 3 and 4 that, for every $i \in N$, there is a collection $\{a_S^i \mid S \in \mathscr{L},\ i \in ex(S)\}$ such that

$$\Phi_i(v) = \sum_{\{S \in \mathscr{L} \mid i \in ex(S)\}} a_S^i [v(S) - v(S \setminus i)],$$

$$\sum_{i \in ex(N)} a_N^i = 1,$$

$$\sum_{i \in ex(S)} a_S^i = \sum_{\{j \notin S \mid S \cup j \in \mathscr{L}\}} a_{S \cup j}^j \quad \forall S \in \mathscr{L},\ S \neq \emptyset,\ S \neq N.$$

Therefore, it suffices to show that

$$a_S^i = \frac{c(S \setminus i)\, c([S, N])}{c(N)} \quad \text{for all } S \in \mathscr{L} \text{ and } i \in ex(S).$$

Note that the coefficients are $a_S^i = \Phi_i(\delta_S)$, hence the chain axiom implies that $a_S^j c(S \setminus i)$

$= a_S^i c(S \setminus j)$ for all $i, j \in ex(S)$. If we fix $i \in ex(S)$, then we obtain

$$\sum_{j \in ex(S)} a_S^j = a_S^i + \sum_{\{j \in ex(S) \mid j \neq i\}} \frac{c(S \setminus j)}{c(S \setminus i)} a_S^i$$
$$= \frac{a_S^i}{c(S \setminus i)} \sum_{j \in ex(S)} c(S \setminus j)$$
$$= a_S^i \frac{c(S)}{c(S \setminus i)}.$$

For $S = N$ we take the first efficiency equation and hence $c(N \setminus i) = a_N^i c(N)$, for every $i \in ex(N)$. Thus,

$$a_N^i = \frac{c(N \setminus i)\, c([N, N])}{c(N)} \quad \forall i \in ex(N).$$

We assume the following induction hypothesis: For every $T \in \mathscr{L}$, $|T| = k \geqslant 2$ we have

$$a_T^i = \frac{c(T \setminus i)\, c([T, N])}{c(N)} \quad \forall i \in ex(T).$$

The case $k = n$, that is $T = N$ has just been proved. Let $S \in \mathscr{L}$, such that $|S| = k - 1 < n$. Then $S \neq N$ and the efficiency equations imply that

$$\sum_{i \in ex(S)} a_S^i = \sum_{\{j \notin S \mid S \cup j \in \mathscr{L}\}} a_{S \cup j}^j$$
$$= \sum_{\{j \notin S \mid S \cup j \in \mathscr{L}\}} \frac{c(S)\, c([S \cup j, N])}{c(N)}$$
$$= \frac{c(S)}{c(N)} \sum_{\{j \notin S \mid S \cup j \in \mathscr{L}\}} c([S \cup j, N])$$
$$= \frac{c(S) c([S, N])}{c(N)},$$

where the second equality follows from the induction hypothesis for $T = S \cup j$. Finally, for every $i \in ex(S)$, the identity

$$a_S^i \frac{c(S)}{c(S \setminus i)} = \frac{c(S) c([S, N])}{c(N)} \quad \text{implies}$$
$$a_S^i = \frac{c(S \setminus i) c([S, N])}{c(N)}. \qquad \square$$

Remark 3. Note that if $\mathscr{L} = 2^N$ then $ex(S) = S$ for all coalition $S \subseteq N$, and

$$\frac{c(S \setminus i) c([S, N])}{c(N)} = \frac{(s-1)!(n-s)!}{n!}.$$

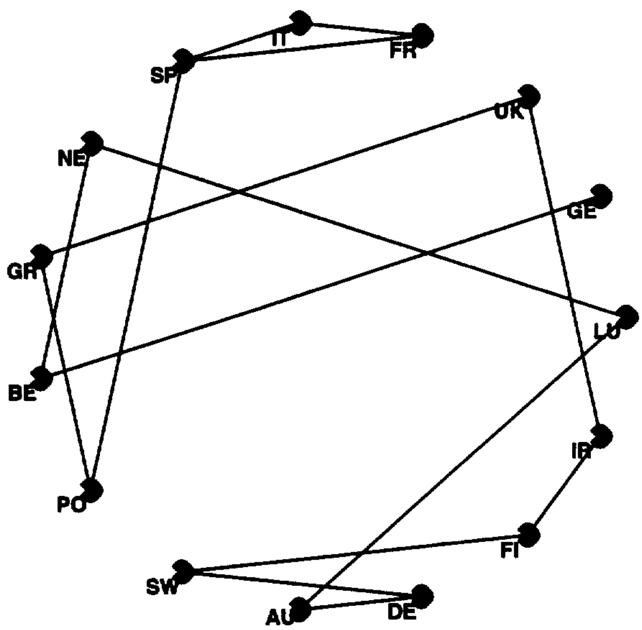
4. Application to European Union

We study the voting power in the Council of Ministers of the European Union, which is the main decision making body in the EU. Herne and Nurmi [8] have studied the power indices (Shapley–Shubik and Banzhaf) in the EU Council. Their analyses are based on the qualified majority requirement (70%) and there is no restrictions to cooperation between countries.

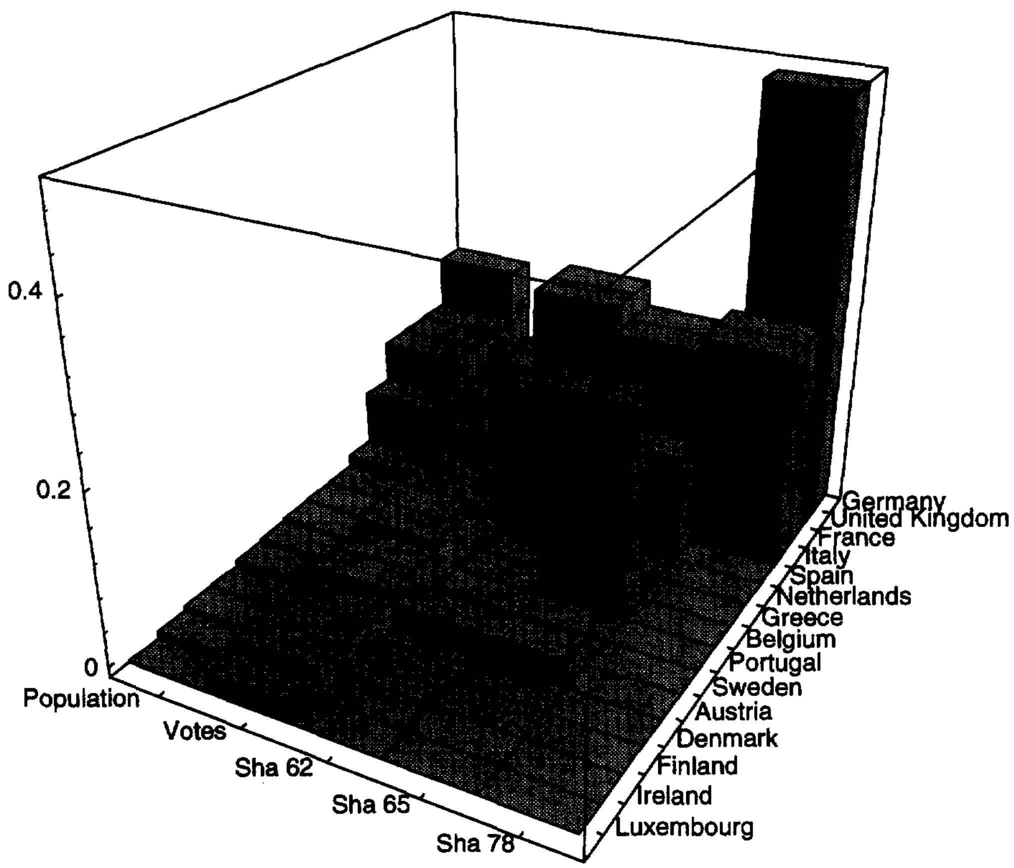
The power index employed is the Shapley value for games on convex geometries. Furthermore, we consider the convex geometry of the connected subgraphs of a communication situation defined by the next graph. There is a block containing the following countries {FR, IT, SP} and a chain

{GE, BE, NE, LU, AU, DE,
SW, FI, IR, UK, GR, PO, SP}.

The following figure shows this graph of relations between the 15 countries of the European Union.



Country	Population	Votes	Sha 62	Sha 65	Sha 78
Germany	0.219	0.115	0.125	0.125	0.5
U. K.	0.157	0.115	0	0	0
France	0.156	0.115	0.266	0.234	0.25
Italy	0.155	0.115	0.266	0.234	0.25
Spain	0.106	0.092	0.125	0.125	0
Netherlands	0.041	0.058	0.188	0	0
Greece	0.028	0.058	0	0	0
Belgium	0.027	0.058	0	0.25	0
Portugal	0.027	0.058	0	0	0
Sweden	0.024	0.046	0	0	0
Austria	0.022	0.046	0.031	0.031	0
Denmark	0.014	0.035	0	0	0
Finland	0.014	0.035	0	0	0
Ireland	0.010	0.035	0	0	0
Luxembourg	0.001	0.023	0	0	0



The above Table and 3D-diagram contain the population, votes and Shapley values for the fifteen countries of the European Union. The qualified majority rule is assumed to be 62, 65, and 78 out of 87 votes in the Council of the EU.

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