A SURVEY OF BICOOPERATIVE GAMES

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1. Introduction

The theory of cooperative games studies situations where a group of people/players are associated to obtain a profit as a result of their cooperation. Thus, a cooperative game is defined as a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function verifying that $v(\emptyset) = 0$. For each $S \in 2^N$, the worth $v(S)$ can be interpreted as the maximal gain or minimal cost that the players which form coalition S can achieve themselves against the best offensive threat by the complementary coalition $N \setminus S$. Hence, we can say that a cooperative game has orthogonal coalitions (see Myerson [11]). Classical market games for economies with private goods are examples of cooperative games.

Games with non-orthogonal coalitions are games in which the worth of coalition S are not independent of the actions of coalition $N \setminus S$. Clearly, social situations involving externalities and public goods are such cases. For instance, we consider a group of agents with a common good which is causing them expenses or costs. In a external or internal way, a modification (sale, buying, etc.) of this good is proposed to them. This action will suppose a greater profit to them in case they all agree with the change proposed about the actual situation of the good. Moreover, even though the patrimonial good can be divisible, we suppose that the greatest value of the selling operation is reached if we consider all the common good.

A possibility of modeling these situations may be the following. We consider pairs (S, T), with $S, T \subseteq N$ and $S \cap T = \emptyset$. Thus, (S, T) is a partition of N in three groups. Players in S are defenders of modifying the *status quo* and they want to accept a proposal; players in T do not agree with modifying the situation and they will take action against any change. Finally, the members of $N \setminus (S \cup T)$ are not convinced of the profits derived from the proposal and they vote abstention.

Thus, in our model we consider the set of all ordered pairs of disjoint coalitions $3^N = \{(S, T) : S, T \subseteq N, \ \ S \cap T = \emptyset\}$, and define a function $b : 3^N \to \mathbb{R}$. For each $(S, T) \in 3^N$, the worth $b(S, T)$ can be interpreted as the maximal gain (whenever $b(S, T) > 0$) or minimal loss (whenever $b(S, T) < 0$) that the players of the coalition S can achieve when they decide to play together against the players of T and the players of $N \setminus (S \cup T)$ not taking part. This leads us in a natural way into the concept of bicooperative game introduced by Bilbao [1].

Definition 1. *A bicooperative game is a pair (N, b) with N a finite set and b is a function $b : 3^N \to \mathbb{R}$ with $b(\emptyset, \emptyset) = 0$.*

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Similarly to the cooperative case in which each coalition $S \in 2^N$ can be identified with a $\{0,1\}$-vector $\mathbf{1}_S$, each pair $(S,T) \in 3^N$ can be identified with the $\{-1,0,1\}$-vector $\mathbf{1}_{(S,T)}$ defined, for all $i \in N$, by

$$\mathbf{1}_{(S,T)}(i) = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

An especial kind of bicooperative games has been studied by Felsenthal and Machover [5] who consider *ternary voting games*. This concept is a generalization of voting games which recognizes abstention as an option alongside *yes* and *no* votes. These games are given by mappings $u : 3^N \to \{-1,1\}$ satisfying the following three conditions: $u(N,\emptyset) = 1$, $u(\emptyset,N) = -1$, and $\mathbf{1}_{(S,T)}(i) \leq \mathbf{1}_{(S',T')}(i)$ for all $i \in N$, implies $u(S,T) \leq u(S',T')$. A negative outcome, -1, is interpreted as defeat and a positive outcome, 1, as passage of a bill.

In Chua and Huang [3] the Shapley-Shubik index for ternary voting games is considered. More recently, several works by Freixas [6, 7] and Freixas and Zwicker [8] have been devoted to the study of voting systems with several ordered levels of approval in the input and in the output. In their model, the abstention is a level of input approval intermediate between *yes* and *no* votes. A new approach to bicooperative games is presented by Grabisch and Lange [10] by using the product of finite distributive lattices. They consider a finite set of players $N = \{1,\ldots,n\}$ and the product of lattices $(L_1,\leq_1) \times \cdots \times (L_n,\leq_n)$ such that every $L_i = \{-1,0.1\}$, where 1 means voting or playing in favor, -1 means voting or playing against, and 0 means abstention.

A one-point solution concept for cooperative games is a function which assigns to every cooperative game a n-dimensional real vector which represents a payoff distribution over the players. The study of solution concepts is central in cooperative game theory. The most important solution concept is the *Shapley value* as proposed by Shapley [13]. The Shapley value assumes that every player is equally likely to join to any coalition of the same size and all coalitions with the same size are equally likely. The Shapley value $\Phi(v) \in \mathbb{R}^n$ of game v is a weighted average of the marginal contributions of the players and for player $i \in N$, it is given by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!\,(n-1-s)!}{n!} \left[v(S \cup \{i\}) - v(S)\right],$$

where $s = |S|$ and $n = |N|$.

Another form to introduce the Shapley value is based in the marginal worth vectors and corresponds to the following interpretation. Suppose the players enter a room one by one in a randomly chosen order. Each player gets the amount that he contributes to the coalition S already formed into the room when the player i enters the room; that is, i gets $v(S \cup \{i\}) - v(S)$. The Shapley value $\Phi(v)$ distributes to each player $i \in N$, the expected amount that he gets by this procedure, that is,

$$\Phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi_n} \left[v\left(\pi^i \cup \{i\}\right) - v\left(\pi^i\right)\right].$$

where Π_n is the set of all permutations of N and π^i is the set of the predecessors of player i in the order π.

A solution concept for cooperative games is a function which assigns to every cooperative game (N, v) with $|N| = n$, a subset of n-dimensional real vectors which represent the payoff distribution over the players. The core is one of the most studied solution concepts. The *core* of a cooperative game (N, v) consists of all payoff vectors which distribute the total savings $v(N)$ among players and secure to every coalition $S \in 2^N$ at least the amount it can obtain by operating on its own, that is,

$$C(N,v) = \left\{x \in \mathbb{R}^n : x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \in 2^N\right\},$$

where $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$.

Although it is considered a very natural solution concept, has the trouble that, in many cases, it is empty. For the class of convex games [14], it can be affirmed that it is a nonempty set; nevertheless, even though the core is not empty, it could be small for obtaining reasonable solutions to certain games. This leads to consider other solution concepts. In 1978, Weber [16] proposed as a solution concept for a cooperative game, a set that contains the core, is always nonempty and easier to compute. Its definition is based in the marginal worth vectors. Each permutation of the elements of N, $\pi = (i_1, i_2, \ldots, i_n)$, can be interpreted as a sequential process of formation of the grand coalition N. Beginning from the empty set, first the player i_1 is incorporated, next the player i_2 and so until the incorporation of the player i_n give rise to the coalition N. In each one of these processes, the corresponding *marginal worth vector*, $a^\pi(v) \in \mathbb{R}^n$, evaluates the marginal contribution of every player to the coalition formed by his predecessors, that is,

$$a_i^\pi(v) = v\left(\pi^i \cup \{i\}\right) - v\left(\pi^i\right) \text{ for all } i \in N,$$

where π^i is the set of the predecessors of player i in the order π. The *Weber set* of game v is the convex hull of all marginal worth vectors, that is,

$$W(N,v) = conv\left\{a^\pi(v) : \pi \in \Pi_n\right\}.$$

Let us outline the contents of our work. In the next section, we study some properties and characteristics of the distributive lattice 3^N. The aim of the third section is to introduce the Shapley value for a bicooperative game. We obtain an axiomatization of the Shapley value in this context as well as a nice formula to compute it. This value is the only one that satisfies our five axioms. Four of them are extensions of the classical axioms for the Shapley value: linearity, symmetry, dummy and efficiency. The fifth axiom is refereed to the structure of the family of signed coalitions. In the fourth section we define the above solutions concepts for bicooperative games and prove that the core is always contained in the Weber set. In the relation between the Weber set and the core, the bisupermodular games, which are defined in the fifth section, play an important role. We see that the bisupermodular games are the only ones for which their Weber set and the core coincide, establishing a characterization of these games. Throughout this chapter, we will write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$ respectively.

2. The lattice 3^N

Let $N = \{1, \ldots, n\}$ be a finite set and let $3^N = \{(A, B) : A, B \subseteq N,\ \ A \cap B = \emptyset\}$. Grabisch and Labreuche [9] proposed a relation in 3^N given by

$$(A, B) \sqsubseteq (C, D) \Longleftrightarrow A \subseteq C,\ B \supseteq D.$$

The set $\left(3^N, \sqsubseteq\right)$ is a partially ordered set (or poset) with the following properties:
1. $(\emptyset, N)$ is the first element: $(\emptyset, N) \sqsubseteq (A, B)$ for all $(A, B) \in 3^N$.
2. $(N, \emptyset)$ is the last element: $(A, B) \sqsubseteq (N, \emptyset)$ for all $(A, B) \in 3^N$.
3. Every pair of elements of 3^N has a join

$$(A, B) \vee (C, D) = (A \cup C, B \cap D),$$

and a meet

$$(A, B) \wedge (C, D) = (A \cap C, B \cup D).$$

Moreover, $\left(3^N, \sqsubseteq\right)$ is a finite distributive lattice. Two pairs (A, B) and (C, D) are comparable if $(A, B) \sqsubseteq (C, D)$ or $(C, D) \sqsubseteq (A, B)$; otherwise, (A, B) and (C, D) are incomparable. A chain of 3^N is an induced subposet of 3^N in which any two elements are comparable. In $\left(3^N, \sqsubseteq\right)$, all maximal chains have the same number of elements and this number is $2n + 1$. Thus, we can consider the rank function

$$\rho : 3^N \to \{0, 1, \ldots, 2n\}$$

such that $\rho\left[(\emptyset, N)\right] = 0$ and $\rho\left[(S, T)\right] = \rho\left[(A, B)\right] + 1$ if (S, T) covers (A, B), that is, if $(A, B) \sqsubset (S, T)$ and there no exists $(H, J) \in 3^N$ such that $(A, B) \sqsubset (H, J) \sqsubset (S, T)$.

For the distributive lattice 3^N, let P denote the set of all nonzero $\vee$-irreducible elements. Then P is the disjoint union $C_1 + C_2 + \cdots + C_n$ of the chains

$$C_i = \{(\emptyset, N \setminus i),\ (i, N \setminus i)\}, \quad 1 \leq i \leq n = |N|.$$

An order ideal of P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in I$. The set of all order ideals of P, ordered by inclusion, is the distributive lattice $J(P)$, where the lattice operations $\vee$ and $\wedge$ are just ordinary union and intersection. The fundamental theorem for finite distributive lattices (see [15, Theorem 3.4.1]) states that the map $\varphi : 3^N \to J(P)$ given by $(A, B) \mapsto \{(X, Y) \in P : (X, Y) \sqsubseteq (A, B)\}$ is an isomorphism (see Figure 1).

Example. Let $N = \{1, 2\}$. Then $P = \{(\emptyset, \{1\}), (\emptyset, \{2\}), (\{2\}, \{1\}), (\{1\}, \{2\})\}$ is the disjoint union of the chains $(\emptyset, \{1\}) \sqsubset (\{2\}, \{1\})$ and $(\emptyset, \{2\}) \sqsubset (\{1\}, \{2\})$. We will denote $a = (\emptyset, \{1\}),\ b = (\{2\}, \{1\}),\ c = (\emptyset, \{2\}),\ d = (\{1\}, \{2\})\}$, and hence

$$J(P) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}.$$

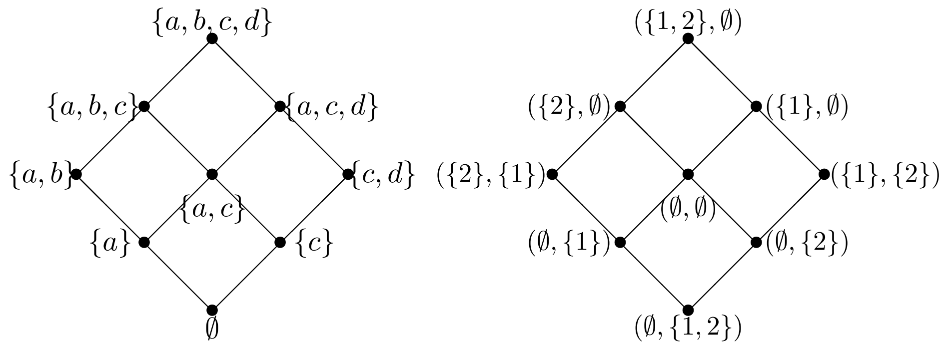

Fig. 1

In the following, we will denote by $c\left(3^N\right)$ the number of maximal chains in 3^N and by $c\left(\left[(A,B),(C,D)\right]\right)$ the number of maximal chains in the sublattice $\left[(A,B),(C,D)\right]$.

Proposition 1. *The number of maximal chains of 3^N is $(2n)!/2^n$, where $n = |N|$.*

Proof. The number of maximal chains of 3^N is equal to the number of maximal chains of $J(P)$ and this number is also equal to the number of extensions $e(P)$ of P to a total order (see Stanley [15, Section 3.5]).

Since $P = C_1 + \cdots + C_n$, where the chain C_i satisfies $|C_i| = 2$ for $1 \le i \le n$, we can apply the enumeration of lattice paths method from Stanley [15, Example 3.5.4], and obtain

$$c\left(3^N\right) = e(P) = \binom{2n}{2, \ldots, 2} = \frac{(2n)!}{2^n}. \quad \square$$

Proposition 2. *For all $(A,B) \in 3^N$, the number of maximal chains of the sublattice $\left[(\emptyset, N), (A,B)\right]$ is $(n+a-b)!/2^a$, where $a = |A|$ and $b = |B|$.*

Proof. Given the sublattice $\left[(\emptyset, N), (A,B)\right]$, we consider $N \setminus B = \{i_1, \ldots, i_{n-b}\}$ and hence there are $n-b$ elements $(\emptyset, N \setminus i)$ with $i \notin B$ (see Figure 2).

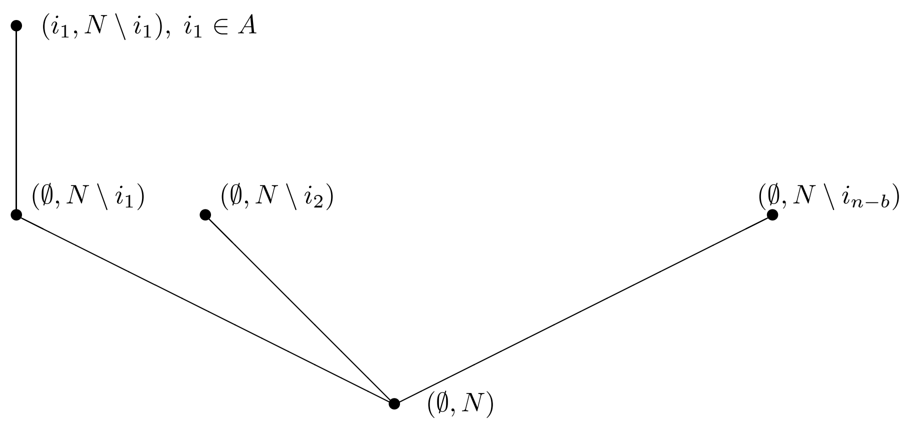

Fig. 2

Since $A \subseteq N \setminus B$, then $a \leq n - b$ and thus, the set of the irreducible elements of the sublattice can be written as

$$P_{[(\emptyset,N),(A,B)]} = C_1 + \cdots + C_a + C_{a+1} + \cdots + C_{a+(n-b-a)}$$

where for all $i_j \in A$, $1 \leq j \leq a$ and $i_{a+k} \notin A \cup B$, $1 \leq k \leq n-b-a$, we obtain

$$\begin{aligned} C_j &= \{(\emptyset, N \setminus i_j), (i_j, N \setminus i_j)\}, \\ C_{a+k} &= \{(\emptyset, N \setminus i_{a+k})\}. \end{aligned}$$

That is, there are a chains such that $|C_j| = 2$ and there are $n-b-a$ chains such that $|C_{a+k}| = 1$. Since

$$|C_1| + \cdots + |C_a| + |C_{a+1}| + \cdots + \left|C_{a+(n-b-a)}\right| = 2a + (n-b-a),$$

we can apply the enumeration of lattice paths method from Stanley [15, Section 3.5] and we obtain

$$c([(\emptyset, N), (A,B)]) = \binom{2a + (n-b-a)}{2, \ldots, 2, 1, \ldots, 1} = \frac{(n+a-b)!}{2^a}. \quad \square$$

Proposition 3. *Let* $(A,B), (C,D) \in 3^N$ *with* $(A,B) \sqsubseteq (C,D)$. *The number of maximal chains of the sublattice* $[(A,B),(C,D)]$ *is equal to the number of maximal chains of the sublattice* $[(D,C),(B,A)]$.

Proof. First of all, note that if $(A,B) \sqsubseteq (C,D)$, then $A \subseteq C$, $B \supseteq D$ and hence $(D,C) \sqsubseteq (B,A)$. Therefore, $[(D,C),(B,A)]$ is a sublattice of 3^N.

Let $\varphi : (3^N, \sqsubseteq) \to (3^N, \sqsubseteq)$ be the map defined by $\varphi(A,B) = (B,A)$. This map is one to one since

$$\varphi(A,B) = \varphi(C,D) \Longleftrightarrow (B,A) = (D,C) \Longleftrightarrow B = D,\ A = C \Longleftrightarrow (A,B) = (C,D).$$

Clearly, if $(A,B) \sqsubset (A_1,B_1) \sqsubset \cdots \sqsubset (A_k,B_k) \sqsubset (C,D)$ is a maximal chain in the sublattice $[(A,B),(C,D)]$ then

$$(D,C) \sqsubset (B_k,A_k) \sqsubset \cdots \sqsubset (B_1,A_1) \sqsubset (B,A)$$

is a maximal chain in the sublattice $[(D,C),(B,A)]$. Finally, it follows that

$$(X,Y) \in [(A,B),(C,D)] \Longleftrightarrow (Y,X) \in [(D,C),(B,A)]. \quad \square$$

3. The Shapley value for bicooperative games

We denote by $\mathcal{BG}^N$ the real vector space of all bicooperative games on N, that is

$$\mathcal{BG}^N = \left\{ b : 3^N \to \mathbb{R},\ b(\emptyset,\emptyset) = 0 \right\}.$$

We consider the *identity games* $\left\{\delta_{(S,T)} : (S,T) \in 3^N,\ (S,T) \neq (\emptyset,\emptyset)\right\}$, the *superior unanimity games* $\left\{\overline{u}_{(S,T)} : (S,T) \in 3^N,\ (S,T) \neq (\emptyset,\emptyset)\right\}$ and the *inferior unanimity games* $\left\{\underline{u}_{(S,T)} : (S,T) \in 3^N,\ (S,T) \neq (\emptyset,\emptyset)\right\}$, which are defined, for any $(S,T) \in 3^N$ such that $(S,T) \neq (\emptyset,\emptyset)$ as follows.

The identity game $\delta_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\delta_{(S,T)}(A,B) = \begin{cases} 1 & \text{if } (A,B) = (S,T), \\ 0 & \text{otherwise.} \end{cases}$$

The superior unanimity game $\overline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is given by

$$\overline{u}_{(S,T)}(A,B) = \begin{cases} 1 & \text{if } (S,T) \sqsubseteq (A,B), \quad (A,B) \neq (\emptyset,\emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

The inferior unanimity game $\underline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\underline{u}_{(S,T)}(A,B) = \begin{cases} -1 & \text{if } (A,B) \sqsubseteq (S,T), \quad (A,B) \neq (\emptyset,\emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to prove (see [2]) that all the above collections are bases of $\mathcal{BG}^N$.

A *value* on $\mathcal{BG}^N$ is a function $\Phi : \mathcal{BG}^N \to \mathbb{R}^n$, which associates to each bicooperative game b a vector $(\Phi_1(b), \ldots, \Phi_n(b))$ which represents the value that every player has in the game b. In order to define a reasonable value for a bicooperative game and following the same issue and interpretation of the Shapley value in the cooperative case, we consider that a player i estimates his participation in game b, evaluating his marginal contributions $b(S \cup i, T) - b(S,T)$ in those signed coalitions $(S \cup i, T)$ that are formed from others (S,T) when i is incorporated to S and his marginal contributions $b(S,T) - b(S, T \cup i)$ in those (S,T) that are formed when i leaves the coalition $T \cup i$.

Thus, a value for player i can be written as

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}^i_{(S,T)} \left(b(S \cup i, T) - b(S,T) \right) + \underline{p}^i_{(S,T)} \left(b(S,T) - b(S, T \cup i) \right) \right],$$

where for every (S,T), the coefficient $\overline{p}^i_{(S,T)}$ can be interpreted as the subjective probability that the player i has of joining the coalition S and $\underline{p}^i_{(S,T)}$ as the subjective probability that the player i has of leaving the coalition $T \cup i$. Thus, $\Phi_i(b)$ is the value that the player i can expect in the game b.

Figure 3 shows the different sequential orders corresponding to the different chains from $(\emptyset, N)$ to $(N, \emptyset)$ which contain (S,T) and $(S \cup i, T)$ and all chains that contain the signed coalitions $(S, T \cup i)$ and (S,T).

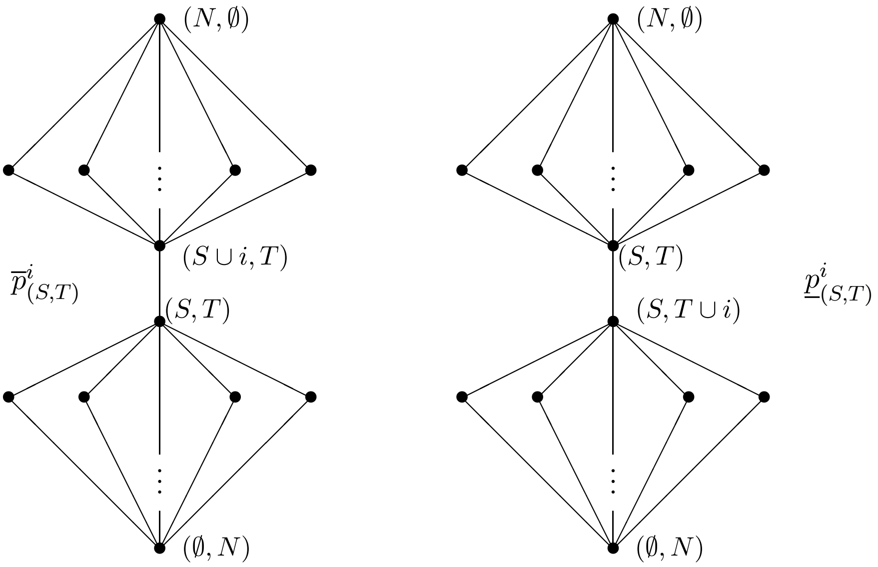

Fig. 3

If we assume that all sequential orders or chains have the same probability, we can deduce formulas for these probabilities $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ in terms of the number of chains which contain to these coalitions. Applying Propositions 2 and 3, we obtain

$$\begin{aligned}\overline{p}^i_{(S,T)} &= \frac{c\left(\left[\left(\emptyset,N\right),\left(S,T\right)\right]\right)\ c\left(\left[\left(S\cup i,T\right),\left(N,\emptyset\right)\right]\right)}{c\left(3^N\right)} \\ &= \frac{\dfrac{(n+s-t)!}{2^s}\cdot\dfrac{(n+t-s-1)!}{2^t}}{\dfrac{(2n)!}{2^n}} \\ &= \frac{(n+s-t)!\,(n+t-s-1)!}{(2n)!}2^{n-s-t},\end{aligned}$$

$$\begin{aligned}\underline{p}^i_{(S,T)} &= \frac{c\left(\left[\left(\emptyset,N\right),\left(S,T\cup i\right)\right]\right)\ c\left(\left[\left(S,T\right)\left(N,\emptyset\right)\right]\right)}{c\left(3^N\right)} \\ &= \frac{\dfrac{(n+t-s)!}{2^t}\cdot\dfrac{(n+s-t-1)!}{2^s}}{\dfrac{(2n)!}{2^n}} \\ &= \frac{(n+t-s)!\,(n+s-t-1)!}{(2n)!}2^{n-s-t}.\end{aligned}$$

Taking into account that $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ are independent of player i, and only depend of $s=|S|$ and $t=|T|$, we can establish the following definition.

Definition 2. *The Shapley value for the bicooperative game $b\in\mathcal{BG}^N$ is given, for each $i\in N$, by*

$$\Phi_i(b)=\sum_{(S,T)\in 3^{N\setminus i}}\left[\overline{p}_{s,t}\left(b(S\cup i,T)-b(S,T)\right)+\underline{p}_{s,t}\left(b(S,T)-b\left(S,T\cup i\right)\right)\right],$$

where, for all $(S,T) \in 3^{N\backslash i}$,

$$\overline{p}_{s,t} = \frac{(n+s-t)!\,(n+t-s-1)!}{(2n)!} 2^{n-s-t}$$

and

$$\underline{p}_{s,t} = \frac{(n+t-s)!\,(n+s-t-1)!}{(2n)!} 2^{n-s-t}.$$

With the aim to characterize the Shapley value for bicooperative games, we consider a set of reasonable axioms and we prove that the Shapley value is the unique value on $\mathcal{BG}^N$ which satisfies these axioms.

Linearity axiom. For all $\alpha, \beta \in \mathbb{R}$, and $b, w \in \mathcal{BG}^N$,

$$\Phi_i(\alpha b + \beta w) = \alpha \Phi_i(b) + \beta \Phi_i(w).$$

We now introduce the dummy axiom, understanding that a player is a *dummy player* when his contributions to signed coalitions $(S \cup i, T)$ formed with his incorporation to S and his contributions to signed coalitions (S,T) formed with his desertion of $T \cup i$ coincide exactly with his individual contributions, that is, a player $i \in N$ is a dummy in $b \in \mathcal{BG}^N$ if, for every $(S,T) \in 3^{N\backslash i}$, it holds

$$\begin{aligned} b(S \cup i, T) - b(S,T)) &= b(\{i\}, \emptyset), \\ b(S,T) - b(S, T \cup i) &= -b(\emptyset, \{i\}). \end{aligned}$$

Note that if $i \in N$ is a dummy in $b \in \mathcal{BG}^N$ then, for all $(S,T) \in 3^{N\backslash i}$,

$$b(S \cup i, T) - b(S, T \cup i) = b(\{i\}, \emptyset) - b(\emptyset, \{i\}).$$

Since a dummy player i in a game b has no meaningful strategic role in the game, the value that this player should expect in the game b must exactly be the sum up of his contributions.

Dummy axiom. If player $i \in N$ is dummy in $b \in \mathcal{BG}^N$, then

$$\Phi_i(b) = b(\{i\}, \emptyset) - b(\emptyset, \{i\}).$$

In the similar way to the cooperative case, for the comparison of roles in a game to be meaningful, the evaluation of a particular position should depend on the structure of the game but not on the labels of the players.

Symmetry axiom. For all $b \in \mathcal{BG}^N$ and for any permutation π over N, it holds that $\Phi_{\pi i}(\pi b) = \Phi_i(b)$ for all $i \in N$, where $\pi b(\pi S, \pi T) = b(S,T)$ and $\pi S = \{\pi i : i \in S\}$.

In a cooperative game, it is assumed that all players decide to cooperate among them and form the grand coalition N. This leads to the problem of distributing the amount $v(N)$ among them. Taking into account different situations that can be modelled by a bicooperative game b, we suppose that the amount $b(N, \emptyset)$ is the maximal gain and $b(\emptyset, N)$ is the minimal loss obtained by the players when they decide full cooperation. Then the maximal global gain is given by $b(N, \emptyset) - b(\emptyset, N)$. From this perspective, the value Φ must satisfy the following axiom.

Efficiency axiom. For every $b \in \mathcal{BG}^N$, it holds

$$\sum_{i \in N} \Phi_i(b) = b(N, \emptyset) - b(\emptyset, N).$$

It is easy to check that our Shapley value for bicooperative games verifies the above axioms. But this value is not the unique value which satisfies these four axioms. For instance, the value $\Phi(b)$ defined, for $b \in \mathcal{BG}^N$ and $i \in N$, by

$$\Phi_i(b) = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} \left[b\left(S \cup i, N \setminus (S \cup i)\right) - b\left(S, N \setminus S\right) \right],$$

also verifies these axioms. However, note that, for any bicooperative game $b \in \mathcal{BG}^N$, this value is the Shapley value corresponding to the cooperative game (N, v), where $v : 2^N \to \mathbb{R}$ is defined by $v(A) = b(A, N \setminus A)$ if $A \neq \emptyset$, and $v(\emptyset) = 0$. This value is not satisfactory for any bicooperative game in the sense that only consider the contributions to signed coalitions in which all players take part. Moreover, there is an infinity of different bicooperative games which give rise to the same cooperative game.

For these reasons, if we want to obtain an axiomatic characterization of our Shapley value for bicooperative games, we need to introduce an additional axiom. Previously, we show that a value on $\mathcal{BG}^N$ that satisfy the above four axioms is given by the expression

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}_{s,t} \left(b(S \cup i, T) - b(S,T) \right) + \underline{p}_{s,t} \left(b(S,T) - b\left(S, T \cup i\right) \right) \right],$$

where $\overline{p}_{s,t}$ and $\underline{p}_{s,t}$ satisfy some conditions. We prove this result in several steps. First of all, we show that a value for player i satisfying the linearity and dummy axioms can be expressed as a linear combination of his contributions.

Theorem 4. *Let Φ_i be a value for player $i \in N$ which satisfies linearity and dummy axioms. Then, for every $b \in \mathcal{BG}^N$,*

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}^i_{(S,T)} \left(b(S \cup i, T) - b(S,T) \right) + \underline{p}^i_{(S,T)} \left(b(S,T) - b\left(S, T \cup i\right) \right) \right],$$

where $\displaystyle\sum_{(S,T) \in 3^{N \setminus i}} \overline{p}^i_{(S,T)} = 1$, *and* $\displaystyle\sum_{(S,T) \in 3^{N \setminus i}} \underline{p}^i_{(S,T)} = 1$.

Proof. The set of identity games is a basis of $\mathcal{BG}^N$, and each game $b \in \mathcal{BG}^N$ can be written as

$$b = \sum_{\{(S,T) \in 3^N : (S,T) \neq (\emptyset,\emptyset)\}} b(S,T)\delta_{(S,T)}.$$

By the linearity axiom,

$$\Phi_i(b) = \sum_{\{(S,T) \in 3^N : (S,T) \neq (\emptyset,\emptyset)\}} \Phi_i(\delta_{(S,T)}) b(S,T).$$

We denote by $a^i_{(S,T)} = \Phi_i\left(\delta_{(S,T)}\right)$ for all $(S,T) \neq (\emptyset,\emptyset)$ and thus, the value $\Phi_i(b)$ is given by

$$\begin{aligned}
&\sum_{(S,T)\in 3^N} a^i_{(S,T)} b(S,T) \\
= &\sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S,T)} b(S,T) + \sum_{\{(S,T)\in 3^N : i\in S\}} a^i_{(S,T)} b(S,T) + \sum_{\{(S,T)\in 3^N : i\in T\}} a^i_{(S,T)} b(S,T) \\
= &\sum_{\left\{(S,T)\in 3^{N\setminus i} : (S,T)\neq(\emptyset,\emptyset)\right\}} a^i_{(S,T)} b(S,T) + \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S\cup i,T)} b(S\cup i,T) \\
&+ \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S,T\cup i)} b(S,T\cup i) \\
= &\sum_{\left\{(S,T)\in 3^{N\setminus i} : (S,T)\neq(\emptyset,\emptyset)\right\}} \left(a^i_{(S,T)} b(S,T) + a^i_{(S\cup i,T)} b(S\cup i,T) + a^i_{(S,T\cup i)} b(S,T\cup i)\right) \\
&+ a^i_{(\{i\},\emptyset)} b(\{i\},\emptyset) + a^i_{(\emptyset,\{i\})} b(\emptyset,\{i\}).
\end{aligned}$$

Let us consider the games $w^i_{(A,B)} : 3^N \to \mathbb{R}$ where, for each $(A,B) \in 3^{N\setminus i}$, the game $w^i_{(A,B)}$ is defined by

$$w^i_{(A,B)}(S,T) = \begin{cases} w^i_{(A,B)}(S\setminus i,T) & \text{if } i\in S, \\ w^i_{(A,B)}(S,T\setminus i) & \text{if } i\in T, \\ 1 & \text{if } i\notin S\cup T, (\emptyset,\emptyset)\neq(S,T)\sqsubseteq(A,B), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, player i is a dummy in $w^i_{(A,B)}$ for each $(A,B)\in 3^{N\setminus i}$ and hence $\Phi_i(w^i_{(A,B)}) = 0$ by the dummy axiom. If we apply the above equality to the game $w^i_{(A,B)}$ we get

$$\sum_{\left\{(S,T)\in 3^{N\setminus i} : (\emptyset,\emptyset)\neq(S,T)\sqsubseteq(A,B)\right\}} \left(a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)}\right) = 0.$$

We show, by induction on $\rho\left[(S,T)\right]$, the rank of the signed coalitions, that for all $(S,T)\in 3^{N\setminus i}$, $(S,T)\neq(\emptyset,\emptyset)$, it holds that $a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} = 0$. Note that the first element in $\left(3^{N\setminus i},\sqsubseteq\right)$ is $(\emptyset, N\setminus i)$, and so $\rho\left[(\emptyset,N\setminus i)\right] = 0$. Hence

$$\sum_{\left\{(S,T)\in 3^{N\setminus i} : (S,T)\sqsubseteq(\emptyset,N\setminus i)\right\}} \left(a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)}\right) = a^i_{(\emptyset,N\setminus i)} + a^i_{(\{i\},N\setminus i)} + a^i_{(\emptyset,N)} = 0.$$

Now assume the property for $(H,J)\in 3^{N\setminus i}$ with $\rho\left[(H,J)\right] \leq k-1$ and suppose that $(S,T)\in 3^{N\setminus i}$ has $\rho\left[(S,T)\right] = k$. Then

$$\begin{aligned}
\Phi_i(w^i_{(S,T)}) &= \sum_{\left\{(H,J)\in 3^{N\setminus i} : (\emptyset,\emptyset)\neq(H,J)\sqsubseteq(S,T)\right\}} \left(a^i_{(H,J)} + a^i_{(H\cup i,J)} + a^i_{(H,J\cup i)}\right) \\
&= a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} \\
&\quad + \sum_{\left\{(H,J)\in 3^{N\setminus i} : (\emptyset,\emptyset)\neq(H,J)\sqsubset(S,T)\right\}} \left(a^i_{(H,J)} + a^i_{(H\cup i,J)} + a^i_{(H,J\cup i)}\right) \\
&= a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} = 0,
\end{aligned}$$

where the last but one equality follows from the induction hypothesis, and the last one follows from the dummy axiom. Now for each $(S,T) \in 3^{N\setminus i}$, define

$$\overline{p}^i_{(\emptyset,\emptyset)} = a^i_{(\{i\},\emptyset)}, \quad \underline{p}^i_{(\emptyset,\emptyset)} = -a^i_{(\emptyset,\{i\})}, \quad \overline{p}^i_{(S,T)} = a^i_{(S\cup i,T)}, \quad \underline{p}^i_{(S,T)} = -a^i_{(S,T\cup i)},$$

and we compute

$$\begin{aligned}\Phi_i(b) &= \sum_{(S,T)\in 3^{N\setminus i}} \left[\left(\underline{p}^i_{(S,T)} - \overline{p}^i_{(S,T)}\right) b(S,T) + \overline{p}^i_{(S,T)} b(S\cup i,T) - \underline{p}^i_{(S,T)} b\left(S,T\cup i\right)\right] \\ &= \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)} \left(b(S\cup i,T) - b(S,T)\right) + \underline{p}^i_{(S,T)} \left(b(S,T) - b\left(S,T\cup i\right)\right)\right].\end{aligned}$$

Finally, it is easy to check that player i is a dummy in the games $\overline{u}_{(\{i\},N\setminus i)}$ and $\underline{u}_{(N\setminus i,\{i\})}$, and hence

$$\begin{aligned}\sum_{(S,T)\in 3^{N\setminus i}} \overline{p}^i_{(S,T)} &= \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S\cup i,T)} = \sum_{\{(S,T)\in 3^N : i\in S\}} a^i_{(S,T)} \\ &= \sum_{\{(S,T)\in 3^N : i\in S\}} \Phi_i\left(\delta_{(S,T)}\right) = \Phi_i\left(\sum_{\{(S,T)\in 3^N : i\in S\}} \delta_{(S,T)}\right) \\ &= \Phi_i\left(\overline{u}_{(\{i\},N\setminus i)}\right) = \overline{u}_{(\{i\},N\setminus i)}\left(\{i\},\emptyset\right) - \overline{u}_{(\{i\},N\setminus i)}\left(\emptyset,\{i\}\right) = 1.\end{aligned}$$

$$\begin{aligned}\sum_{(S,T)\in 3^{N\setminus i}} \underline{p}^i_{(S,T)} &= \sum_{(S,T)\in 3^{N\setminus i}} -a^i_{(S,T\cup i)} = \sum_{\{(S,T)\in 3^N : i\in T\}} -a^i_{(S,T)} \\ &= \sum_{\{(S,T)\in 3^N : i\in T\}} -\Phi_i\left(\delta_{(S,T)}\right) = \Phi_i\left(\sum_{\{(S,T)\in 3^N : i\in T\}} -\delta_{(S,T)}\right) \\ &= \Phi_i\left(\underline{u}_{(N\setminus i,\{i\})}\right) = \underline{u}_{(N\setminus i,\{i\})}\left(\{i\},\emptyset\right) - \underline{u}_{(N\setminus i,\{i\})}\left(\emptyset,\{i\}\right) = 1. \quad \square\end{aligned}$$

Now, we show that if add the symmetry axiom to the linearity and dummy axioms, the coefficients $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ only depend of the cardinality of S and T.

Theorem 5. *Let Φ_i be a value for player $i \in N$ defined, for every game $b \in \mathcal{BG}^N$, by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)} \left(b(S\cup i,T) - b(S,T)\right) + \underline{p}^i_{(S,T)} \left(b(S,T) - b\left(S,T\cup i\right)\right)\right].$$

If Φ_i satisfies the symmetry axiom, then $\overline{p}^i_{(S,T)} = \overline{p}_{s,t}$ and $\underline{p}^i_{(S,T)} = \underline{p}_{s,t}$ for all $(S,T) \in 3^{N\setminus i}$ with $s = |S|$ and $t = |T|$.

Proof. Let Φ_i be a value for player i given by

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)} \left(b(S\cup i,T) - b(S,T)\right) + \underline{p}^i_{(S,T)} \left(b(S,T) - b\left(S,T\cup i\right)\right)\right].$$

Let (S_1, T_1) and (S_2, T_2) be signed coalitions in $3^{N\setminus i}$ such that $(S_1, T_1) \neq (\emptyset, \emptyset) \neq (S_2, T_2)$ satisfying that $|S_1| = |S_2| < n-1$ and $|T_1| = |T_2| < n-1$. Consider a permutation π of N that takes $\pi S_1 = S_2$ and $\pi T_1 = T_2$ while leaving i fixed. Then $\pi\delta_{(S_1,T_1)} = \delta_{(S_2,T_2)}$ and

$$\begin{aligned} \overline{p}^i_{(S_1,T_1)} &= \Phi_i(\delta_{(S_1\cup i,T_1)}) = \Phi_i(\delta_{(S_2\cup i,T_2)}) = \overline{p}^i_{(S_2,T_2)}, \\ \underline{p}^i_{(S_1,T_1)} &= -\Phi_i(\delta_{(S_1,T_1\cup i)}) = -\Phi_i(\delta_{(S_2,T_2\cup i)}) = \underline{p}^i_{(S_2,T_2)}, \end{aligned}$$

where the second equality follows from the symmetry axiom.

Now, let $i, j \in N, i \neq j$ and let $(S,T) \in 3^{N\setminus\{i,j\}}$. Let us consider the permutation π of N that interchanges i and j while leaving the remaining players fixed. Then $\pi\delta_{(S,T)} = \delta_{(S,T)}$ and

$$\begin{aligned} \overline{p}^i_{(S,T)} &= \Phi_i(\delta_{(S\cup i,T)}) = \Phi_j(\delta_{(S\cup j,T)}) = \overline{p}^j_{(S,T)}, \\ \underline{p}^i_{(S,T)} &= -\Phi_i(\delta_{(S,T\cup i)}) = -\Phi_j(\delta_{(S,T\cup j)}) = \underline{p}^j_{(S,T)}. \end{aligned}$$

Moreover,

$$\begin{aligned} \overline{p}^i_{(N\setminus i,\emptyset)} &= \Phi_i(\delta_{(N,\emptyset)}) = \Phi_j(\delta_{(N,\emptyset)}) = \overline{p}^j_{(N\setminus j,\emptyset)}, \\ \underline{p}^i_{(\emptyset,N\setminus i)} &= -\Phi_i(\delta_{(\emptyset,N)}) = -\Phi_j(\delta_{(\emptyset,N)}) = \underline{p}^j_{(\emptyset,N\setminus j)}. \end{aligned}$$

Hence, for every $(S,T) \in 3^{N\setminus i}$ there exist $\overline{p}_{s,t}$ and $\underline{p}_{s,t}$ such that $\overline{p}^i_{(S,T)} = \overline{p}_{s,t}$ and $\underline{p}^i_{(S,T)} = \underline{p}_{s,t}$ for all $i \in N$. □

The following theorem characterizes the values $\Phi = (\Phi_1, \ldots, \Phi_n)$ which satisfy the above axioms and are efficient.

Theorem 6. *Let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be a value on $\mathcal{BG}^N$ defined, for every game b and for all $i \in N$, by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \Big[\overline{p}_{s,t}\,(b(S\cup i,T) - b(S,T)) + \underline{p}_{s,t}\,(b(S,T) - b(S,T\cup i))\Big].$$

Then, the value Φ satisfies the efficiency axiom if and only if it is satisfied

$$\overline{p}_{n-1,0} = \frac{1}{n}, \quad \underline{p}_{0,n-1} = \frac{1}{n},$$

and

$$(n-s-t)\,\overline{p}_{s,t} + t\underline{p}_{s,t-1} = (n-s-t)\,\underline{p}_{s,t} + s\overline{p}_{s-1,t}$$

for all $0 \leq s,\, t \leq n-1$ and $0 < s+t \leq n-1$.

Proof. For every $b \in \mathcal{BG}^N$ we have that $\sum_{i \in N} \Phi_i(b)$ is equal to

$$\begin{aligned}
& \sum_{i \in N} \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}_{s,t} \left(b(S \cup i, T) - b(S,T) \right) + \underline{p}_{s,t} \left(b(S,T) - b(S, T \cup i) \right) \right] \\
= & \sum_{i \in N} \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}_{s,t} b(S \cup i, T) - \underline{p}_{s,t} b(S, T \cup i) + \left(-\overline{p}_{s,t} + \underline{p}_{s,t} \right) b\left(S,T\right) \right] \\
= & \sum_{(S,T) \in 3^N} b\left(S,T\right) \left[s\overline{p}_{s-1,t} - t\underline{p}_{s,t-1} + (n-s-t) \left(-\overline{p}_{s,t} + \underline{p}_{s,t} \right) \right] \\
= & \; b\left(N, \emptyset\right) n\overline{p}_{n-1,0} - b\left(\emptyset, N\right) n\underline{p}_{0,n-1} \\
& + \sum_{\substack{(S,T) \in 3^N \\ (S,T) \notin \{(\emptyset,\emptyset),(\emptyset,N),(N,\emptyset)\}}} b\left(S,T\right) \left[s\overline{p}_{s-1,t} - t\underline{p}_{s,t-1} + (n-s-t) \left(-\overline{p}_{s,t} + \underline{p}_{s,t} \right) \right].
\end{aligned}$$

If the coefficients satisfy the relations for the coefficients, then Φ satisfies the efficiency axiom.

Conversely, fix $(S,T) \in 3^N, (S,T) \neq (\emptyset, \emptyset)$, and applying the preceding equality to the identity game $\delta_{(S,T)}$, we have

$$\sum_{i \in N} \Phi_i(\delta_{(S,T)}) = \begin{cases} n\overline{p}_{n-1,0} & \text{if } (S,T) = (N, \emptyset), \\ -n\underline{p}_{0,n-1} & \text{if } (S,T) = (\emptyset, N), \\ s\overline{p}_{s-1,t} - t\underline{p}_{s,t-1} + (n-s-t) \left(\underline{p}_{s,t} - \overline{p}_{s,t} \right) & \text{otherwise.} \end{cases}$$

Thus, if Φ satisfies the efficiency axiom, the relations for the coefficients are true. □

As we have already indicated, these four axioms are not sufficient to characterize the Shapley value for bicooperative games. Now, we introduce an additional axiom and prove that our Shapley value is the unique value on $\mathcal{BG}^N$ that verifies the five axioms. This new axiom will take into account the structure of the set of the signed coalitions.

First of all, note that the signed coalitions $(S \setminus j, T)$ and $(S, T \cup i)$ where $j \in S$ and $i \notin S \cup T$ have the same rank

$$\rho\left[(S \setminus j, T)\right] = \rho\left[(S, T \cup i)\right] = n + s - t - 1.$$

However, the number of maximal chains in the sublattice $\left[(\emptyset, N), (S \setminus j, T)\right]$ is not the same that the number of maximal chains in $\left[(\emptyset, N), (S, T \cup i)\right]$ since, by Proposition 2,

$$\begin{aligned}
c\left(\left[(\emptyset, N), (S \setminus j, T)\right]\right) &= \frac{(n+s-1-t)!}{2^{s-1}}, \\
c\left(\left[(\emptyset, N), (S, T \cup i)\right]\right) &= \frac{(n+s-t-1)!}{2^{s}}.
\end{aligned}$$

Hence, beginning from the signed coalition $(\emptyset, N)$, the probability of formation of the signed coalition (S,T) with the incorporation of one player j to $(S \setminus j, T)$ must be distinct to the probability of formation (S,T) with the desertion of one player i in $(S,T\cup i)$.

In analogous form, if we consider $(S,T\setminus k)$ with $k\in T$ and $(S\cup i,T)$ which have the same rank, the number of maximal chains in $[(S,T\setminus k),(N,\emptyset)]$ is not equal to number of maximal chains in $[(S\cup i,T),(N,\emptyset)]$. Therefore the probability of formation of $(N,\emptyset)$ beginning from $(S,T\setminus k)$ when one player k leaves the coalition T must be distinct to the probability of formation of $(N,\emptyset)$ when one player i form the signed coalition $(S\cup i,T)$.

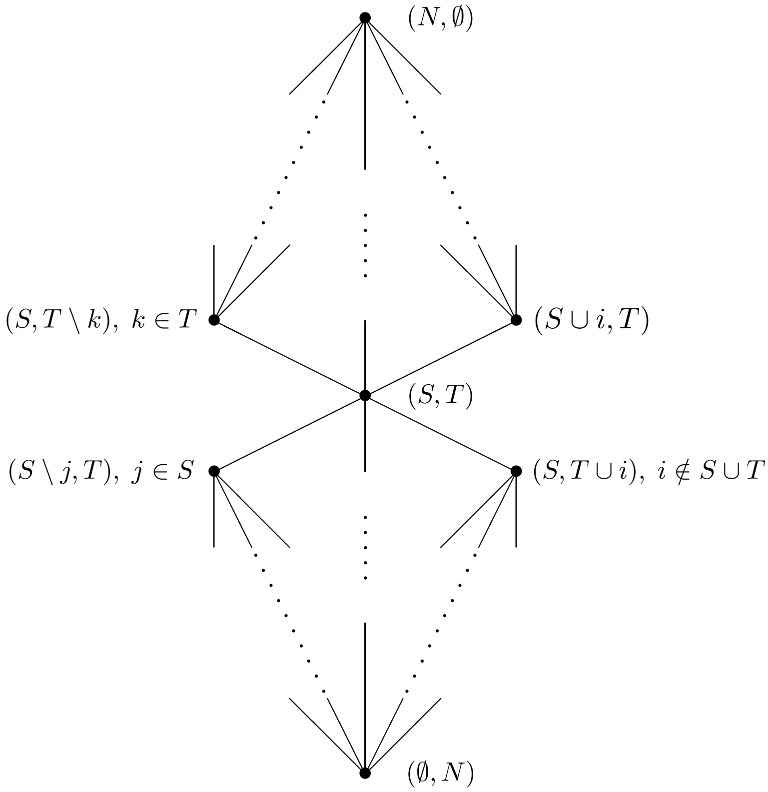

Fig. 4

Taking into account these considerations, the values that one player must obtain in the identity games must be proportional to the number of maximal chains in the corresponding sublattices. It must be also considered that one value verifying the above four axioms assigns a non-negative real number to one player i in the identity game $\delta_{(S,T)}$ if this player belongs S and a non-positive real number if the player i belongs T. From this point of view, our value must be satisfied the following axiom (see Figure 4).

Structural axiom. For every $(S,T)\in 3^{N\setminus i}$, $j\in S$ and $k\in T$, it holds

$$\frac{c\left([(\emptyset,N),(S\setminus j,T)]\right)}{c\left([(\emptyset,N),(S,T\cup i)]\right)}=-\frac{\Phi_j(\delta_{(S,T)})}{\Phi_i\left(\delta_{(S,T\cup i)}\right)},\quad \frac{c\left([(S,T\setminus k),(N,\emptyset)]\right)}{c\left([(S\cup i,T),(N,\emptyset)]\right)}=-\frac{\Phi_k(\delta_{(S,T)})}{\Phi_i\left(\delta_{(S\cup i,T)}\right)}.$$

Theorem 7. *Let Φ be a value on $\mathcal{BG}^N$. The value Φ is the Shapley value if and only if Φ satisfies the efficiency axiom and each component satisfies linearity, dummy, symmetry and structural axioms.*

Proof. If Φ is a value that satisfies linearity, dummy, symmetry and efficiency, then

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}_{s,t}\left(b(S\cup i,T) - b(S,T)\right) + \underline{p}_{s,t}\left(b(S,T) - b\left(S,T\cup i\right)\right)\right]$$

and the coefficients $\overline{p}_{s,t}$ and $\underline{p}_{s,t}$ satisfy

$$\overline{p}_{n-1,0} = \frac{1}{n}, \quad \underline{p}_{0,n-1} = \frac{1}{n},$$

and

$$(n-s-t)\,\overline{p}_{s,t} + t\underline{p}_{s,t-1} = (n-s-t)\,\underline{p}_{s,t} + s\overline{p}_{s-1,t}. \tag{1}$$

Taking into account that the value Φ verifies the structural axiom then

$$\overline{p}_{s-1,t} = 2\underline{p}_{s,t} \tag{2}$$

$$\underline{p}_{s,t-1} = 2\overline{p}_{s,t} \tag{3}$$

We prove that these coefficients, verifying all above conditions, are determined in unique form. Indeed, consider a coalition (S,T) with $|S| = n-1$ and $|T| = 0$. If we apply the equation 1 to this coalition, we obtain

$$\overline{p}_{n-1,0} = \underline{p}_{n-1,0} + (n-1)\,\overline{p}_{n-2,0}$$

and by 2, $\overline{p}_{n-2,0} = 2\underline{p}_{n-1,0}$. Taking into account that $\overline{p}_{n-1,0} = \frac{1}{n}$ and combining the above equalities, we have that

$$\frac{1}{n} = \left(1 + 2\,(n-1)\right)\underline{p}_{n-1,0}$$

and hence

$$\underline{p}_{n-1,0} = \frac{1}{n\,(2n-1)} = \frac{1!\,(2n-2)!}{2^{n-1}\,(2n)!}2^n, \quad \overline{p}_{n-2,0} = \frac{2}{n\,(2n-1)} = \frac{1!\,(2n-2)!}{2^{n-2}\,(2n)!}2^n.$$

In similar way, if we apply 1 and 2 to a signed coalition (S,T) with $|S| = n-2$ and $|T| = 0$, we get

$$\begin{aligned} 2\overline{p}_{n-2,0} &= 2\underline{p}_{n-2,0} + (n-2)\,\overline{p}_{n-3,0}, \\ \overline{p}_{n-3,0} &= 2\underline{p}_{n-2,0}, \end{aligned}$$

and hence

$$\underline{p}_{n-2,0} = \frac{2!\,(2n-3)!}{2^{n-2}\,(2n)!}2^n, \quad \overline{p}_{n-3,0} = \frac{2!\,(2n-3)!}{2^{n-3}\,(2n)!}2^n.$$

If we assume that

$$\underline{p}_{s+1,0} = \frac{(n-s-1)!\,(n+s)!}{2^{s+1}\,(2n)!}2^n, \quad \overline{p}_{s,0} = \frac{(n-s-1)!\,(n+s)!}{2^{s}\,(2n)!}2^n$$

then, for $|S| = s$ and $|T| = 0$, applying 1 and 2,

$$\begin{aligned} (n-s)\,\overline{p}_{s,0} &= (n-s)\,\underline{p}_{s,0} + s\overline{p}_{s-1,0}, \\ \overline{p}_{s-1,0} &= 2\underline{p}_{s,0}, \end{aligned}$$

and combining both expressions, we obtain, for $1 \leq s \leq n-1$,

$$\underline{p}_{s,0} = \frac{(n-s)!\,(n+s-1)!}{2^{s}\,(2n)!}2^n, \quad \overline{p}_{s-1,0} = \frac{(n-s)!\,(n+s-1)!}{2^{s-1}\,(2n)!}2^n.$$

If we apply the same reasoning with the equalities 1 and 3 beginning with a coalition (S,T) with $|S| = 0$ and $|T| = n-1$, we obtain, for $1 \leq t \leq n-1$,

$$\overline{p}_{0,t} = \frac{(n-t)!\,(n+t-1)!}{2^{t}\,(2n)!}2^n, \quad \underline{p}_{0,t-1} = \frac{(n-t)!\,(n+t-1)!}{2^{t-1}\,(2n)!}2^n.$$

If we now consider (S,T) with $|S| = s$ and $|T| = 1$, we apply 1 and 3,

$$(n-s-1)\,\overline{p}_{s,1} + \underline{p}_{s,0} = (n-s-1)\,\underline{p}_{s,1} + s\overline{p}_{s-1,1},$$

$$\overline{p}_{s,1} = \frac{1}{2}\underline{p}_{s,0}, \quad \overline{p}_{s-1,1} = \frac{1}{2}\underline{p}_{s-1,0},$$

and substitute the values already obtained, then

$$\overline{p}_{s-1,1} = \frac{(n-s+1)!\,(n+s-2)!}{2^{s}\,(2n)!}2^n, \quad \underline{p}_{s,1} = \frac{(n-s+1)!\,(n+s-2)!}{2^{s+1}\,(2n)!}2^n.$$

If we assume that

$$\overline{p}_{s-1,t-1} = \frac{(n-s+t-1)!\,(n+s-t)!}{2^{s+t-2}\,(2n)!}2^n, \quad \underline{p}_{s,t-1} = \frac{(n-s+t-1)!\,(n+s-t)!}{2^{s+t-1}\,(2n)!}2^n,$$

then applying $\underline{p}_{s,t-1} = 2\overline{p}_{s,t}$ (3) we obtain, for all $0 \leq s,\ \ t \leq n-1$ and $s+t \leq n-1$,

$$\overline{p}_{s,t} = \frac{(n+s-t)!\,(n+t-s-1)!}{2^{s+t}\,(2n)!}2^n.$$

Finally, applying 1 and 2,

$$\begin{aligned} (n-s-t)\,\overline{p}_{s,t} + t\underline{p}_{s,t-1} &= (n-s-t)\,\underline{p}_{s,t} + s\overline{p}_{s-1,t}, \\ \overline{p}_{s-1,t} &= 2\underline{p}_{s,t}, \end{aligned}$$

it holds that

$$\underline{p}_{s,t} = \frac{(n+t-s)!\,(n+s-t-1)!}{2^{s+t}\,(2n)!}2^n$$

for all $0 \leq s,\ \ t \leq n-1$ and $s+t \leq n-1$. $\square$

4. The core and the Weber set

Now, some solution concepts for bicooperative games are introduced, understanding as a solution concept any subset of vectors in $\mathbb{R}^n$ that provide an equitable distribution of the total saving among the players. Taking into account different situations that can be modelled by a bicooperative game (N,b), the amount $b(N,\emptyset)$ is the maximal gain and $b(\emptyset,N)$ is the minimal loss obtained by the players when they decide full cooperation and so, the maximal global gain is given by $b(N,\emptyset)-b(\emptyset,N)$. A vector $x\in\mathbb{R}^n$ which satisfies $\sum_{i\in N}x_i=b(N,\emptyset)-b(\emptyset,N)$ is called *efficient vector* and the set of all efficient vectors is called *preimputation set* which is defined by

$$I^*(N,b)=\left\{x\in\mathbb{R}^n:\sum_{i\in N}x_i=b(N,\emptyset)-b(\emptyset,N)\right\}.$$

The *imputations* for game b are the preimputations that satisfy the *individual rationality principle* for all players, that is, each player gets at least the difference between the amount that he can attain for himself taking the rest of players against and the value of the signed coalition $(\emptyset,N)$,

$$I(N,b)=\{x\in I^*(N,b):x_i\geq b(i,N\setminus i)-b(\emptyset,N)\ \text{ for all } i\in N\}.$$

A satisfactory distribution criterion could be that every signed coalition $(S,T)\in 3^N$ receives at least the amount it can contribute to the coalition $(\emptyset,N)$, that is, the amount $b(S,T)-b(\emptyset,N)$. It leads us to define the notion of the core of the game b as the set

$$C(N,b)=\left\{\begin{array}{l} x\in I^*(N,b):\ x=y+z \text{ with} \\ \qquad\qquad\qquad y(S)+z(N\setminus T)\geq b(S,T)-b(\emptyset,N)\ \ \forall\ (S,T)\in 3^N\end{array}\right\}.$$

This definition can be interpreted in the following manner. For each $(S,T)\in 3^N$, the players who are not in the coalition T have contributed to the formation of (S,T) since they will not act against the player of the coalition S and for this, they must be received a payoff given by the vector z. Moreover, those players of $N\setminus T$ who are in the coalition S must get a different payoff to the rest of players, given by the vector y since these players have contributed to the formation of (S,T) in a different way.

In order to extend the idea of the Weber set to a bicooperative game (N,b), it is assumed that all players estimate that $(N,\emptyset)$ is formed as a sequential process where in each step a different player is incorporated to the *defender coalition* or a different player leaves the *detractor coalition.* These sequential processes are obtained considering the different chains from $(\emptyset,N)$ to $(N,\emptyset)$. In each one of these processes, a player can evaluate his contribution when is incorporated to the defenders or his contribution when leaves the detractors. This can be reflected in the vectors of $\mathbb{R}^n$ denominated *superior marginal worth vectors* and *inferior marginal worth vectors.* With the aim to formalize this idea, we introduce the following notation.

Given $N=\{1,\dots,n\}$, let $\overline{N}=\{-n,\dots,-1,1,\dots,n\}$. We can define an isomorphism $\Lambda:3^N\to 2^{\overline{N}}$ as follows: For each $(S,T)\in 3^N$, $\Lambda(S,T)=S\cup\{-i:i\in N\setminus T\}\in 2^{\overline{N}}$. For instance, $\Lambda(\emptyset,N)=\emptyset$ and $\Lambda(N,\emptyset)=\overline{N}$. Since $S\cap T=\emptyset\Leftrightarrow S\subseteq N\setminus T$ we see that $i\in\Lambda(S,T)$ and $i>0$ imply $-i\in\Lambda(S,T)$.

In the lattice $\left(3^N, \sqsubseteq\right)$, we consider the set of all maximal chains which going from $(\emptyset, N)$ to $(N, \emptyset)$ and denote this set by $\Theta\left(3^N\right)$. If $\theta \in \Theta\left(3^N\right)$ is the maximal chain

$$(\emptyset, N) \sqsubset (S_1, T_1) \sqsubset \cdots \sqsubset (S_j, T_j) \sqsubset \cdots \sqsubset (S_{2n-1}, T_{2n-1}) \sqsubset (N, \emptyset),$$

then we can write the following associated chain of sets in $2^{\overline{N}}$,

$$\emptyset \subset \{i_1\} \subset \cdots \subset \{i_1, \ldots, i_j\} \subset \cdots \subset \{i_1, \ldots, i_{2n-1}\} \subset \overline{N},$$

where $\{i_1, \ldots, i_j\} = \Lambda(S_j, T_j)$ for $j = 1, \ldots, 2n$. We define the vector $\theta(i_j) = (i_1, \ldots, i_j)$, where the last component $i_j \in \overline{N}$ satisfies the following property: if $i_j > 0$ then the player $i_j \in S_j$ and $i_j \notin S_{j-1}$, that is, i_j is the last player who joins S_j and if $i_j < 0$, then the player $-i_j \notin T_j$ and $-i_j \in T_{j-1}$, that is, $-i_j$ is the last player who leaves T_{j-1}. Equivalently, the elements in $\theta(i_j) = (i_1, \ldots, i_j)$ are written following the order of incorporation in the defenders coalitions or desertion of the detractors coalition (depending on the sign of each i_k) in the signed coalitions in chain θ . Moreover, we write

$$\theta(i_j) \setminus i_j = (i_1, i_2, \ldots, i_{j-1}) = \theta(i_{j-1})$$

and $i_k \in \theta(i_j)$ when i_k is one component of the vector $\theta(i_j)$, that is $1 \leq k \leq j$. Note that an equivalence between maximal chains and vectors $\theta = (i_1, \ldots, i_{2n})$ is obtained. Fix an order $\theta = (i_1, \ldots, i_{2n})$, we also define $\alpha[\theta(i_j)] = (S_j, T_j)$ such that $\Lambda(S_j, T_j) = \{i_1, \ldots, i_j\}$. Moreover, $\alpha[\theta(i_j) \setminus i_j] = \alpha[\theta(i_{j-1})] = (S_{j-1}, T_{j-1})$. In particular, $\alpha[\theta(i_{2n})] = (N, \emptyset)$ and $\alpha[\theta(i_1) \setminus i_1] = (\emptyset, N)$.

For example, let $N = \{1, 2, 3\}$ and $\theta \in \Theta\left(3^N\right)$ given by

$$(\emptyset, N) \sqsubset (\emptyset, \{1, 3\}) \sqsubset (\{2\}, \{1, 3\}) \sqsubset (\{2\}, \{1\}) \sqsubset (\{2\}, \emptyset) \sqsubset (\{2, 3\}, \emptyset) \sqsubset (N, \emptyset).$$

Its associated chain of sets in $2^{\overline{N}}$ is given by

$$\emptyset \subset \{-2\} \subset \{-2, 2\} \subset \{-2, 2, -3\} \subset \{-2, 2, -3, -1\} \subset \{-2, 2, -3, -1, 3\} \subset \overline{N}.$$

and the maximal chain can be also represented by the order $\theta = (-2, 2, -3, -1, 3, 1)$. One signed coalition, for instance $(\{2\}, \emptyset)$, can be also represented by $\alpha[\theta(-1)]$ and by $\Lambda^{-1}(\{-2, 2, -3, -1\})$.

Definition 3. *Let $\theta \in \Theta\left(3^N\right)$ and $b \in \mathcal{BG}^N$. We call inferior and superior marginal worth vectors with respect to θ to the vectors $m^\theta(b)$, $M^\theta(b) \in \mathbb{R}^n$ respectively where*

$$\begin{aligned} m_i^\theta(b) &= b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i) \setminus -i]), \\ M_i^\theta(b) &= b(\alpha[\theta(i)]) - b(\alpha[\theta(i) \setminus i]), \end{aligned}$$

for all $i \in N$. We call marginal worth vector with respect to θ to the vector $a^\theta(b) \in \mathbb{R}^n$ obtained as the sum of inferior and superior marginal worth vectors, that is,

$$a_i^\theta(b) = m_i^\theta(b) + M_i^\theta(b), \quad \text{for all } i \in N.$$

The following result show that the marginal worth vectors are preimputations.

Proposition 8. *For any $b \in \mathcal{BG}^N$ and $\theta \in \Theta\left(3^N\right)$ we have*

$$\sum_{i\in N} a_i^\theta(b) = b(N,\emptyset) - b(\emptyset,N).$$

Proof. Let $b \in \mathcal{BG}^N$ and $\theta \in \Theta\left(3^N\right)$. It holds that

$$\begin{aligned}
\sum_{i\in N} a_i^\theta(b) &= \sum_{i\in N}\left[m_i^\theta(b) + M_i^\theta(b)\right] \\
&= \sum_{i\in N}\left[b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i)\setminus -i]) + b(\alpha[\theta(i)]) - b(\alpha[\theta(i)\setminus i])\right] \\
&= \sum_{j=1}^{2n}\left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j)\setminus i_j])\right] \\
&= b(\alpha[\theta(i_1)]) - b(\alpha[\theta(i_1)\setminus i_1]) + \sum_{j=2}^{2n}\left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_{j-1})])\right] \\
&= b(N,\emptyset) - b(\emptyset,N). \quad \square
\end{aligned}$$

Proposition 9. *Let $b \in \mathcal{BG}^N$ and $\theta \in \Theta\left(3^N\right)$. Then,*

$$\sum_{j\in S} M_j^\theta(b) + \sum_{j\in N\setminus T} m_j^\theta(b) = b(S,T) - b(\emptyset,N),$$

for every (S,T) in the chain θ.

Proof. Let $\theta \in \Theta\left(3^N\right)$ and (S,T) in the chain θ with $|S| = s, |T| = t, s+t \le n$ and such that $\Lambda(S,T) = \{i_1, i_2, \ldots, i_{n+s-t}\}$ where the i_j are written following the order of incorporation in θ, that is, $\theta(i_j) = (i_1, i_2, \ldots, i_j)$ for all $1 \le j \le n+s-t$. Then,

$$\begin{aligned}
\sum_{j\in S} M_j^\theta(b) + \sum_{j\in N\setminus T} m_j^\theta(b) &= \sum_{\{i_j\in\Lambda(S,T):i_j>0\}} M_{i_j}^\theta(b) + \sum_{\{i_j\in\Lambda(S,T):i_j<0\}} m_{-i_j}^\theta(b) \\
&= \sum_{i_j\in\Lambda(S,T)}\left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j)\setminus i_j])\right] \\
&= \sum_{j=1}^{n+s-t}\left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j)\setminus i_j])\right] \\
&= b(S,T) - b(\emptyset,N).
\end{aligned}$$

Note that for $(S,T) = (N,\emptyset)$, we have $\sum_{j\in N}\left[m_j^\theta(b) + M_j^\theta(b)\right] = b(N,\emptyset) - b(\emptyset,N)$.

$\square$

Definition 4. *Let $b \in \mathcal{BG}^N$ be a bicooperative game. The Weber set of b is the convex hull of the marginal worth vectors, that is*

$$W(N,b) = conv\left\{a^\theta(b) : \theta \in \Theta\left(3^N\right)\right\}.$$

As the preimputation set is a convex set, it is evident that $W(N,b) \subseteq I^*(N,b)$. However, in general, the vectors of the Weber set are not imputations. For example, let (N,b) with $N = \{1,2\}$ and $b : 3^N \to \mathbb{R}$ defined as

$$b(\emptyset, N) = -5, \quad b(\emptyset, i) = -4, \quad b(i,j) = -1, \quad b(i,\emptyset) = 1, \quad b(N,\emptyset) = 2,$$

for all $i,j \in N$. If we consider $\theta = (-2,2,-1,1)$, then $a_1^\theta(b) = m_1^\theta(b) + M_1^\theta(b) = 3$. As $b(1,2) - b(\emptyset,N) = 4$, then $a_1^\theta(b) < b(1, N\setminus 1) - b(\emptyset, N)$ and $a^\theta(b) \notin I(N,b)$.

It is easy to see, taking into account that $I(N,b)$ is a convex set, that $W(N,b) \subseteq I(N,b)$ if all marginal worth vectors are imputations. For this, a sufficient condition is that the game b is *zero-monotonic*, a concept that is defined as follows.

Definition 5. *A bicooperative game $b \in \mathcal{BG}^N$ is monotonic when for all signed coalitions $(S_1,T_1),(S_2,T_2)$ with $(S_1,T_1) \sqsubseteq (S_2,T_2)$ it holds that $b(S_1,T_1) \le b(S_2,T_2)$.*

Definition 6. *The zero-normalization of a bicooperative game $b \in \mathcal{BG}^N$ is the game $b_0 \in \mathcal{BG}^N$ defined by*

$$b_0(S,T) = b(S,T) - \sum_{j\in S} [b(j, N\setminus j) - b(\emptyset, N)], \quad \text{for all } (S,T) \in 3^N.$$

Definition 7. *A bicooperative game $b \in \mathcal{BG}^N$ is called zero-monotonic if its zero-normalization is monotonic.*

Proposition 10. *Let $b \in \mathcal{BG}^N$ be a zero-monotonic bicooperative game. Then, for every $\theta \in \Theta(3^N)$, the marginal worth vector associated to θ is an imputation for the game b.*

Proof. Let $\theta \in \Theta(3^N)$. Since the vector $a^\theta(b)$ is efficient, we prove that $a_i^\theta(b) \ge b(i, N\setminus i) - b(\emptyset, N)$, for all $i \in N$. Indeed,

$$\begin{aligned}
a_i^\theta(b) &= b(\alpha[\theta(i)]) - b(\alpha[\theta(i)\setminus i]) + b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i)\setminus -i]) \\
&= b_0(\alpha[\theta(i)]) + \sum_{\{i_j \in \theta(i) : i_j > 0\}} [b(i_j, N\setminus i_j) - b(\emptyset, N)] \\
&\quad - b_0(\alpha[\theta(i)\setminus i]) - \sum_{\{i_j \in \theta(i)\setminus i : i_j > 0\}} [b(i_j, N\setminus i_j) - b(\emptyset, N)] \\
&\quad + b_0(\alpha[\theta(-i)]) + \sum_{\{i_j \in \theta(-i) : i_j > 0\}} [b(i_j, N\setminus i_j) - b(\emptyset, N)] \\
&\quad - b_0(\alpha[\theta(-i)\setminus -i]) - \sum_{\{i_j \in \theta(-i)\setminus -i : i_j > 0\}} [b(i_j, N\setminus i_j) - b(\emptyset, N)] \\
&= b_0(\alpha[\theta(i)]) - b_0(\alpha[\theta(i)\setminus i]) + b_0(\alpha[\theta(-i)]) \\
&\quad - b_0(\alpha[\theta(-i)\setminus -i]) + b(i, N\setminus i) - b(\emptyset, N) \\
&\ge b(i, N\setminus i) - b(\emptyset, N),
\end{aligned}$$

where the inequality follows the zero-monotonicity of the bicooperative game b. □

Now we prove that the core of a bicooperative game is always included in its Weber set. It should be noted that the proof of this result is closely related to the proof in [4] of the inclusion of the core in the Weber set for cooperative games.

Theorem 11. *If* $b \in \mathcal{BG}^N$, *then* $C(N,b) \subseteq W(N,b)$.

Proof. Assume that there exists $x \in C(N,b)$ such that $x \notin W(N,b)$. As $x \in C(N,b)$, then $\sum_{i\in N} x_i = b(N,\emptyset) - b(\emptyset,N)$ and $x = y+z$ with $y(S) + z(N\setminus T) \geq b(S,T) - b(\emptyset,N)$ for all $(S,T) \in 3^N$. Since the Weber set $W(N,b)$ is convex and closed we apply the Separation Theorem (see Rockafellar [12]), and so there exists $u \in \mathbb{R}^n$ such that

$$w \cdot u > x \cdot u \text{ for all } w \in W(N,b). \tag{4}$$

In particular for all marginal worth vectors $w = a^\theta(b)$ with $\theta \in \Theta(3^N)$. If the components of vector u are ordered in non increasing order

$$u_{i_1} \geq u_{i_2} \geq \cdots \geq u_{i_{n-1}} \geq u_{i_n},$$

let $\theta \in \Theta(3^N)$ be the maximal chain given by $\theta = (-i_1, i_1, -i_2, i_2, \ldots, -i_n, i_n)$. Note that $\theta(i_j)\setminus i_j = \theta(-i_j)$ for all $1 \leq j \leq n$, $\theta(-i_j)\setminus -i_j = \theta(i_{j-1})$ for all $2 \leq j \leq n$ and $\alpha[\theta(-i_1)\setminus -i_1] = (\emptyset, N)$. Then

$$\begin{aligned}
a^\theta(b)\cdot u &= \sum_{j=1}^{n} a^\theta_{i_j}(b)\,u_{i_j} = \sum_{j=1}^{n}\left[M^\theta_{i_j}(b) + m^\theta_{i_j}(b)\right]u_{i_j} \\
&= \sum_{j=1}^{n} u_{i_j}\left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j)\setminus i_j]) + b(\alpha[\theta(-i_j)]) - b(\alpha[\theta(-i_j)\setminus -i_j])\right] \\
&= \sum_{j=1}^{n} u_{i_j}\left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_{j-1})])\right] \\
&= u_{i_n} b(N,\emptyset) + \sum_{j=1}^{n-1} u_{i_j} b(\alpha[\theta(i_j)]) - u_{i_1} b(\emptyset,N) - \sum_{j=2}^{n} u_{i_j} b(\alpha[\theta(i_{j-1})]) \\
&= u_{i_n} b(N,\emptyset) - u_{i_1} b(\emptyset,N) + \sum_{j=1}^{n-1}\left(u_{i_j} - u_{i_{j+1}}\right) b(\alpha[\theta(i_j)]) \\
&\leq u_{i_n} b(N,\emptyset) - u_{i_1} b(\emptyset,N) + \sum_{j=1}^{n-1}\left(u_{i_j} - u_{i_{j+1}}\right)\left[\sum_{k=1}^{j} y_{i_k} + \sum_{k=1}^{j} z_{i_k} + b(\emptyset,N)\right] \\
&= u_{i_n}\left[\sum_{k=1}^{n} y_{i_k} + \sum_{k=1}^{n} z_{i_k} + b(\emptyset,N)\right] - u_{i_1} b(\emptyset,N) \\
&\quad + \sum_{j=1}^{n-1}\left(u_{i_j} - u_{i_{j+1}}\right)\left[\sum_{k=1}^{j} y_{i_k} + \sum_{k=1}^{j} z_{i_k} + b(\emptyset,N)\right] \\
&= \sum_{j=1}^{n} u_{i_j}\left(y_{i_j} + z_{i_j}\right) = \sum_{j=1}^{n} u_{i_j} x_{i_j} = x\cdot u
\end{aligned}$$

which is in contradiction with the inequality 4. We conclude that a core distribution has to be an element of the Weber set. □

5. Bisupermodular games

Now we introduce a special class of bicooperative games.

Definition 8. *A bicooperative game $b \in \mathcal{BG}^N$ is called bisupermodular if, for all (S_1, T_1) and (S_2, T_2) it holds*

$$b((S_1, T_1) \vee (S_2, T_2)) + b((S_1, T_1) \wedge (S_2, T_2)) \geq b(S_1, T_1) + b(S_2, T_2),$$

or equivalently

$$b(S_1 \cup S_2, T_1 \cap T_2) + b(S_1 \cap S_2, T_1 \cup T_2) \geq b(S_1, T_1) + b(S_2, T_2).$$

The next proposition characterizes the bisupermodular games as those bicooperative games for which the marginal contributions of a player to one signed coalition is never less that the marginal contribution of this player to any signed coalition contained in it. This characterization will be used in the proves of the following results.

Proposition 12. *Let $b \in \mathcal{BG}^N$. The bicooperative game b is bisupermodular if and only if for all $i \in N$ and (S_1, T_1), $(S_2, T_2) \in 3^{N \setminus i}$ such that $(S_1, T_1) \sqsubseteq (S_2, T_2)$, it holds*

$$b(S_2 \cup i, T_2) - b(S_2, T_2) \geq b(S_1 \cup i, T_1) - b(S_1, T_1),$$

and

$$b(S_2, T_2) - b(S_2, T_2 \cup i) \geq b(S_1, T_1) - b(S_1, T_1 \cup i).$$

Proof. *Necessary condition.* Let (S_1, T_1), $(S_2, T_2) \in 3^{N \setminus i}$ with $(S_1, T_1) \sqsubseteq (S_2, T_2)$. If $S_1' = S_1 \cup i$ and we apply the definition of bisupermodularity to (S_1', T_1) and (S_2, T_2), it follows

$$b\left(S_1' \cup S_2, T_1 \cap T_2\right) + b\left(S_1' \cap S_2, T_1 \cup T_2\right) \geq b(S_1 \cup i, T_1) + b(S_2, T_2),$$

and hence

$$b(S_2 \cup i, T_2) + b(S_1, T_1) \geq b(S_1 \cup i, T_1) + b(S_2, T_2).$$

In analogous form, taking $T_2' = T_2 \cup i$ and applying the definition of supermodularity to (S_1, T_1) and (S_2, T_2'), it follows

$$b(S_1, T_1 \cup i) + b(S_2, T_2) \geq b(S_1, T_1) + b(S_2, T_2 \cup i).$$

Sufficient condition. Let (S_1, T_1), $(S_2, T_2) \in 3^N$. If $(S_1, T_1) \sqsubseteq (S_2, T_2)$ or $(S_2, T_2) \sqsubseteq (S_1, T_1)$, the equality trivially holds. So, we consider the case $(S_1, T_1) \wedge (S_2, T_2) \neq (S_1, T_1)$ and $(S_1, T_1) \wedge (S_2, T_2) \neq (S_2, T_2)$.

Let $\theta \in \Theta\left(3^N\right)$ be a maximal chain that contains the signed coalitions (S_2, T_2) and $(S_1, T_1) \vee (S_2, T_2)$. As $\Lambda(S_1, T_1) \setminus \Lambda(S_2, T_2) \neq \emptyset$, we assume that $|\Lambda(S_1, T_1) \setminus \Lambda(S_2, T_2)| = k$ and so, we write $\Lambda(S_1, T_1) \setminus \Lambda(S_2, T_2) = \{i_1, i_2, \ldots, i_k\}$, where the i_j are in the same order that appear in the order θ, i.e.,

$$\alpha[\theta(i_1)] \sqsubset \alpha[\theta(i_2)] \sqsubset \cdots \sqsubset \alpha[\theta(i_k)].$$

Then, the chain θ is given by

$$\emptyset \subset \cdots \subset \Lambda(S_2,T_2) \subset \Lambda(S_2,T_2) \cup \{i_1\} \subset \cdots \subset \Lambda(S_2,T_2) \cup \{i_1,\dots,i_k\} \subset \cdots \subset \overline{N}$$

or equivalently

$$(\emptyset,N) \sqsubset \cdots \sqsubset (S_2,T_2) \sqsubset \cdots \sqsubset (S_1,T_1) \vee (S_2,T_2) \sqsubset \cdots \sqsubset (N,\emptyset).$$

If we denote $A_j = \{i_1,i_2,\dots,i_j\}$, for all $1 \le j \le k$, $A_0 = \emptyset$ and $(P,Q) = (S_1,T_1) \wedge (S_2,T_2)$, it holds that

$\Lambda^{-1}[\Lambda(P,Q) \cup A_j] \sqsubset \Lambda^{-1}[\Lambda(S_2,T_2) \cup A_j]$ for all $1 \le j \le k$. We can apply the hypothesis to $\Lambda^{-1}[\Lambda(P,Q) \cup A_j]$ and $\Lambda^{-1}[\Lambda(S_2,T_2) \cup A_j]$, and we obtain

$$\begin{aligned} & b\left(\Lambda^{-1}(\Lambda(P,Q) \cup A_j)\right) - b\left(\Lambda^{-1}(\Lambda(P,Q) \cup A_{j-1})\right) \\ \le\ & b\left(\Lambda^{-1}(\Lambda(S_2,T_2) \cup A_j)\right) - b\left(\Lambda^{-1}(\Lambda(S_2,T_2) \cup A_{j-1})\right) \end{aligned}$$

for all $1 \le j \le k$. Hence,

$$b((S_1,T_1)) - b((S_1,T_1) \wedge (S_2,T_2)) = b\left(\Lambda^{-1}(\Lambda(P,Q) \cup A_k)\right) - b(P,Q)$$

$$\begin{aligned} &= \sum_{j=1}^{k} \left[b\left(\Lambda^{-1}(\Lambda(P,Q) \cup A_j)\right) - b\left(\Lambda^{-1}(\Lambda(P,Q) \cup A_{j-1})\right)\right] \\ &\le \sum_{j=1}^{k} \left[b\left(\Lambda^{-1}(\Lambda(S_2,T_2) \cup A_j)\right) - b\left(\Lambda^{-1}(\Lambda(S_2,T_2) \cup A_{j-1})\right)\right] \\ &= b((S_1,T_1) \vee (S_2,T_2)) - b(S_2,T_2). \quad \square \end{aligned}$$

The following result permits the identification of the games for which the marginal worth vectors are distributions of the core.

Theorem 13. *A necessary and sufficient condition so that all marginal worth vectors of a bicooperative game $b \in \mathcal{BG}^N$ are vectors of the core is that the game b is bisupermodular*

Proof. *Sufficient condition.* Let $\theta \in \Theta\left(3^N\right)$. We know that the marginal worth vectors are efficient, we prove that the marginal worth vector $a_i^\theta(b) = m_i^\theta(b) + M_i^\theta(b)$ satisfies

$$\sum_{j\in S} M_j^\theta(b) + \sum_{j\in N\setminus T} m_j^\theta(b) \ge b(S,T) - b(\emptyset,N), \quad \text{for all } (S,T) \in 3^N.$$

By Proposition 8, for every (S,T) in the chain θ, it holds

$$\sum_{j\in S} M_j^\theta(b) + \sum_{j\in N\setminus T} m_j^\theta(b) = b(S,T) - b(\emptyset,N).$$

We prove that, for every coalition (S,T), not in the chain θ,

$$\sum_{j\in S} M_j^{\theta}(b) + \sum_{j\in N\setminus T} m_j^{\theta}(b) \geq b(S,T) - b(\emptyset,N).$$

Indeed, let (S,T) be a signed coalition that does not belong to the chain θ, such that $\Lambda(S,T) = \{i_1,i_2,\ldots,i_k\}$, $k = n+s-t$, where the elements are written following the order of θ; that is, $\alpha[\theta(i_1)] \sqsubset \alpha[\theta(i_2)] \sqsubset \cdots \sqsubset \alpha[\theta(i_k)]$.

If we denote $A_j = \{i_1,i_2,\ldots,i_j\}$, for all $1\leq j\leq k$, and $A_0 = \emptyset$, note that, for all $1\leq j\leq k$, we have that $A_j = \Lambda(S,T)\cap\Lambda(\alpha[\theta(i_j)])$, that is $\Lambda^{-1}(A_j) = (S,T)\wedge\alpha[\theta(i_j)]$. As b is a bisupermodular game, the Proposition 12 implies that, for all $1\leq j\leq k$,

$$b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j)\setminus i_j]) \geq b\left(\Lambda^{-1}(A_j)\right) - b\left(\Lambda^{-1}(A_{j-1})\right),$$

and we obtain

$$\begin{aligned}
\sum_{j\in S} M_j^{\theta}(b) + \sum_{j\in N\setminus T} m_j^{\theta}(b) &= \sum_{\{i_j\in\Lambda(S,T):i_j>0\}} M_{i_j}^{\theta}(b) + \sum_{\{i_j\in\Lambda(S,T):i_j<0\}} m_j^{\theta}(b)\\
&= \sum_{i_j\in\Lambda(S,T)} [b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j)\setminus i_j])]\\
&= \sum_{j=1}^{n+s-t} [b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j)\setminus i_j])]\\
&\geq \sum_{j=1}^{n+s-t} \left[b\left(\Lambda^{-1}(A_j)\right) - b\left(\Lambda^{-1}(A_{j-1})\right)\right]\\
&= b(S,T) - b(\emptyset,N).
\end{aligned}$$

Necessary condition. For all $(S_1,T_1),(S_2,T_2)\in 3^N$, consider a maximal chain $\theta\in\Theta\left(3^N\right)$ which contains $(S_1,T_1)\wedge(S_2,T_2) = (S_1\cap S_2, T_1\cup T_2)$ and $(S_1,T_1)\vee(S_2,T_2) = (S_1\cup S_2, T_1\cap T_2)$. As the marginal worth vectors are elements of $C(N,b)$, we have that

$$\sum_{j\in S_1} M_j^{\theta}(b) + \sum_{j\in N\setminus T_1} m_j^{\theta}(b) \geq b(S_1,T_1) - b(\emptyset,N),$$

$$\sum_{j\in S_2} M_j^{\theta}(b) + \sum_{j\in N\setminus T_2} m_j^{\theta}(b) \geq b(S_2,T_2) - b(\emptyset,N),$$

By the election of the maximal chain θ and Proposition 8, it is also verified

$$\sum_{j\in S_1\cap S_2} M_j^{\theta}(b) + \sum_{j\in N\setminus(T_1\cup T_2)} m_j^{\theta}(b) = b((S_1,T_1)\wedge(S_2,T_2)) - b(\emptyset,N).$$

$$\sum_{j\in S_1\cup S_2} M_j^{\theta}(b) + \sum_{j\in N\setminus(T_1\cap T_2)} m_j^{\theta}(b) = b((S_1,T_1)\vee(S_2,T_2)) - b(\emptyset,N).$$

Therefore,

$$b(S_1,T_1) + b(S_2,T_2) - 2b(\emptyset,N)$$

$$
\begin{aligned}
&\leq \sum_{j\in S_1} M_j^{\theta}(b) + \sum_{j\in N\setminus T_1} m_j^{\theta}(b) + \sum_{j\in S_2} M_j^{\theta}(b) + \sum_{j\in N\setminus T_2} m_j^{\theta}(b) \\
&= \sum_{j\in S_1\cup S_2} M_j^{\theta}(b) + \sum_{j\in S_1\cap S_2} M_j^{\theta}(b) + \sum_{j\in N\setminus (T_1\cup T_2)} m_j^{\theta}(b) + \sum_{j\in N\setminus (T_1\cap T_2)} m_j^{\theta}(b) \\
&= b\left((S_1,T_1)\wedge(S_2,T_2)\right) + b\left((S_1,T_1)\vee(S_2,T_2)\right) - 2b(\emptyset,N).
\end{aligned}
$$

Hence

$$
b(S_1,T_1) + b(S_2,T_2) \leq b\left((S_1,T_1)\wedge(S_2,T_2)\right) + b\left((S_1,T_1)\vee(S_2,T_2)\right). \quad \square
$$

As the core of a bicooperative game $b \in \mathcal{BG}^N$ is a convex set, an immediate consequence of this theorem is the following result.

Corollary 14. *Let $b \in \mathcal{BG}^N$. A necessary and sufficient condition so that $W(N,b) = C(N,b)$ is that the bicooperative game b is bisupermodular.*

Note that the Shapley value of a bicooperative game b is given by

$$
\Phi_i(N,b) = \frac{1}{c(3^N)} \sum_{\theta\in\Theta(3^N)} \left[m_i^{\theta}(b) + M_i^{\theta}(b) \right],
$$

for all $i \in N$. Then the Shapley value of a bisupermodular game b is in $C(N,b)$ and hence, the core of a bisupermodular game is non-empty.

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