BICOOPERATIVE GAMES

J. M. BILBAO, J. R. FERNÁNDEZ, A. JIMÉNEZ LOSADA, AND E. LEBRÓN

Matemática Aplicada II, Escuela Superior de Ingenieros
Camino de los Descubrimientos, 41092 Sevilla, Spain
E-mail: mbilbao@cica.es http://www.esi2.us.es/~mbilbao/

Abstract. We will introduce a generalization of the concept of cooperative game. First, we consider the set 3^N of all the ordered pairs of disjoint coalitions. Next, we define bicooperative games $b : 3^N \to \mathbb{R}$ and we study the class of bisubmodular (bisupermodular) games. *Journal of Economic Literature* Classification Number: C71.

1. Introduction

Let $N = \{1, 2, \ldots, n\}$ be a set of players. Aubin [2, Section 13.2] introduced the concept of *generalized coalition* as a function $c : N \to [-1, 1]$ which associates each player i with his/her level of participation $c(i) \in [-1, 1]$. A positive level is interpreted as attraction of the player i for the coalition, and a negative level as repulsion. In particular, the set 2^N is the set of functions $c : N \to \{0, 1\}$.

We will introduce a natural generalization of the concept of cooperative game. First, we consider the set of all the ordered pairs of disjoint coalitions, that is, the set of all *signed coalitions* given by $3^N = \{(S, T) : S, T \subseteq N,\ S \cap T = \emptyset\}$.

Each signed coalition (S, T) of 3^N can be identified with the $\{0, \pm 1\}$-vector $\mathbf{1}_{(S,T)}$ defined by

$$\mathbf{1}_{(S,T)}(i) = \begin{cases} 1, & \text{if } i \in S \\ -1, & \text{if } i \in T \\ 0, & \text{otherwise,} \end{cases}$$

for all $i \in N$. The support of a signed coalition (S, T) of N is $S \cup T$. A partial order $\sqsubseteq$ on 3^N is defined by $(S_1, T_1) \sqsubseteq (S_2, T_2) \iff S_1 \subseteq S_2$ and $T_1 \subseteq T_2$ for all $(S_1, T_1), (S_2, T_2) \in 3^N$.

Example: Felsenthal and Machover [7] introduced *ternary voting games* on a finite set N. This concept is a generalization of voting games which recognizes *abstention* as an option alongside *yes* and *no* votes. These games are given by mappings $u : 3^N \to \{-1, 1\}$ satisfying the following three conditions:

1. $u(N, \emptyset) = 1$,
2. $u(\emptyset, N) = -1$,
3. If $\mathbf{1}_{(S,T)}(i) \leq \mathbf{1}_{(S',T')}(i)$ for all $i \in N$, then $u(S, T) \leq u(S', T')$.

A negative outcome, -1, is interpreted as defeat and a positive outcome, 1, as passage of a bill.

Definition 1.1. *A bicooperative game is a pair (N, b), where N is a finite set and $b : 3^N \to \mathbb{R}$ is a function with $b(\emptyset, \emptyset) = 0$.*

2. Bisubmodular games

We have two binary operations, reduced union $\sqcup$ and intersection $\sqcap$, on 3^N defined as

$$\begin{aligned}(S_1,T_1)\sqcup(S_2,T_2) &= ((S_1\cup S_2)\setminus(T_1\cup T_2),(T_1\cup T_2)\setminus(S_1\cup S_2)),\\ (S_1,T_1)\sqcap(S_2,T_2) &= (S_1\cap S_2,T_1\cap T_2).\end{aligned}$$

Definition 2.1. *A bicooperative game $c:3^N\to\mathbb{R}$ is called bisubmodular if it satisfies $c((S_1,T_1)\sqcup(S_2,T_2))+c((S_1,T_1)\sqcap(S_2,T_2))\le c(S_1,T_1)+c(S_2,T_2)$ for all $(S_1,T_1),(S_2,T_2)\in 3^N$. A bicooperative game $b:3^N\to\mathbb{R}$ is called bisupermodular if $-b$ is bisubmodular and bimodular if the above inequality holds with equality.*

The bisubmodular inequality has been introduced by Chandrasekaran and Kabadi [5]. Fujishige [8, Section 3.5c], Ando and Fujishige [1] studied bisubmodular systems $(\mathcal{F},f)$ where $\mathcal{F}\subseteq 3^N$ is a family of signed subsets closed with respect to the reduced union $\sqcup$ and intersection $\sqcap$, and $f:\mathcal{F}\to\mathbb{R}$ is bisubmodular on $\mathcal{F}$. *Delta-matroids* are essentially equivalent structures considered by Bouchet [3].

The bisubmodular and bisupermodular polyhedra associated with these games are respectively

$$\begin{aligned}P_*(c) &= \left\{x\in\mathbb{R}^n : x(S)-x(T)\le c(S,T) \text{ for all } (S,T)\in 3^N\right\},\\ P_*(b) &= \left\{x\in\mathbb{R}^n : x(S)-x(T)\ge b(S,T) \text{ for all } (S,T)\in 3^N\right\}.\end{aligned}$$

These polyhedra were introduced by Dunstan and Welsh [6] and Bouchet and Cunningham [4] showed that the convex hull of the set of points of a *jump system* is an integral bisubmodular polyhedron.

If $P_*(c)$ is nonempty then $-c(\emptyset,S)\le x(S)\le c(S,\emptyset)$ for all $S\subseteq N$, where $x\in P_*(c)$. Therefore, we obtain $c(\emptyset,S)+c(S,\emptyset)\ge 0$ for all $S\subseteq N$. Fujishige [8, Theorem 3.63] showed that this necessary condition is also sufficient.

We remark that if $c:3^N\to\mathbb{R}$ is a bisubmodular game then

$$c(\emptyset,S)+c(S,\emptyset)\ge 2c(\emptyset,\emptyset)=0,$$

and this implies that every bisubmodular polyhedron is nonempty. The same holds for the class of bisupermodular polyhedra.

A pair $(S,T)\in 3^N$ such that its support $S\cup T=N$ is called an *orthant*. The nonempty face of $P_*(c)$ defined by

$$B_{(S,T)}(c)=\{x\in P_*(c): x(S)-x(T)=c(S,T)\}$$

is called the base polyhedron of c in the orthant (S,T). Moreover, Fujishige [8, Section 3.5b] showed that the set of extreme points (vertices) of $P_*(c)$ satisfies

$$\operatorname{ex}(P_*(c))=\bigcup\left\{\operatorname{ex}\left(B_{(S,T)}(c)\right):\ (S,T) \text{ is an orthant}\right\}.$$

Proposition 2.1. *Let $c:3^N\to\mathbb{R}$ be a bisubmodular game, let (S,T) be an orthant, and let $c_{ST}:2^N\to\mathbb{R}$ be the cooperative game given by $c_{ST}(X)=c(S\cap X,T\cap X)$ for all $X\subseteq N$. Then the game c_{ST} is concave.*

Proof. The bisubmodular inequality implies that for all $X,Y\subseteq N$,

$$\begin{aligned}c_{ST}(X\cup Y)+c_{ST}(X\cap Y) &= c(S_1\cup S_2,T_1\cup T_2)+c(S_1\cap S_2,T_1\cap T_2)\\ &= c((S_1,T_1)\sqcup(S_2,T_2))+c((S_1,T_1)\sqcap(S_2,T_2))\\ &\le c(S_1,T_1)+c(S_2,T_2)\\ &= c_{ST}(X)+c_{ST}(Y),\end{aligned}$$

where $S_1 = S \cap X,\ S_2 = S \cap Y,\ T_1 = T \cap X,\ T_2 = T \cap Y$. Hence the cooperative game c_{ST} is concave. ∎

Note that if b is a bisupermodular game and (S,T) an orthant, then the cooperative game b_{ST} is convex.

Proposition 2.2. *Let $v : 2^N \to \mathbb{R}$ be a convex game, and let $b : 3^N \to \mathbb{R}$ be the bicooperative game given by $b(S,T) := v(S) - v^*(T) = v(S) + v(N \setminus T) - v(N)$. Then b is bisupermodular. Moreover, $P_*(b) = Core(v)$.*

Proof. We compute for all $(S_1,T_1), (S_2,T_2) \in 3^N$,

$$\begin{aligned} b((S_1,T_1) \sqcup (S_2,T_2)) &= v(S \setminus T) + v(N \setminus (T \setminus S)) - v(N) \\ &= v(S \cap (N \setminus T)) + v(S \cup (N \setminus T)) - v(N) \\ &\geq v(S) + v(N \setminus T) - v(N), \end{aligned}$$

where $S = S_1 \cup S_2$ and $T = T_1 \cup T_2$. Thus

$$\begin{aligned} b((S_1,T_1) \sqcup (S_2,T_2)) &\geq v(S_1 \cup S_2) + v((N \setminus T_1) \cap (N \setminus T_2)) - v(N), \\ b((S_1,T_1) \sqcap (S_2,T_2)) &= v(S_1 \cap S_2) + v((N \setminus T_1) \cup (N \setminus T_2)) - v(N). \end{aligned}$$

Since v is supermodular, i.e., $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, the sum satisfies

$$b((S_1,T_1) \sqcup (S_2,T_2)) + b((S_1,T_1) \sqcap (S_2,T_2)) \geq b(S_1,T_1) + b(S_2,T_2)$$

and the bicooperative game b is bisupermodular. Furthermore, the core of the profit game (N,v) is the bisupermodular polyhedron associated with b, that is, $P_*(b) = \{x \in \mathbb{R}^n : x(S) \geq v(S) \text{ for all } S \subseteq N,\ x(N) = v(N)\}$. ∎

References

[1] Ando, K. and S. Fujishige (1996): On structures of bisubmodular polyhedra, *Math. Programming* 74, 293–317.

[2] Aubin, J. P. (1993): *Optima and equilibria: an introduction to nonlinear analysis*, Springer-Verlag, Berlin.

[3] Bouchet, A. (1987): Greedy algorithm and symmetric matroids, *Math. Programming* 38, 147–159.

[4] Bouchet, A. and W. H. Cunningham (1995): Delta-matroids, jump systems and bisubmodular polyhedra, *SIAM J. Disc. Math.* 8, 17–32.

[5] Chandrasekaran, R. and S. N. Kabadi (1988): Pseudomatroids, *Discrete Math.* 71, 205–217.

[6] Dunstan, F. D. J. and D. J. A. Welsh (1973): A greedy algorithm for solving a certain class of linear programmes, *Math. Programming* 5, 338–353.

[7] Felsenthal, D. S. and M. Machover (1997): Ternary voting games, *Int. J. Game Theory* 26, 335–351.

[8] Fujishige, S. (1991): *Submodular Functions and Optimization,* North-Holland, Amsterdam.