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The Shapley value for bicooperative games

J.M. Bilbao · J.R. Fernández · N. Jiménez · J.J. López

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Abstract The aim of the present paper is to study a one-point solution concept for bicooperative games. For these games introduced by Bilbao (Cooperative Games on Combinatorial Structures, 2000), we define a one-point solution called the Shapley value, since this value can be interpreted in a similar way to the classical Shapley value for cooperative games. The main result of the paper is an axiomatic characterization of this value.

Keywords Bicooperative game · Shapley value

1 Introduction

A cooperative game is defined as a pair (N, v), where N is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a function verifying that $v(\emptyset) = 0$. For each $S \in 2^N$, the worth $v(S)$ can be interpreted as the maximal gain or minimal cost that the players which form the coalition S can achieve themselves against the best offensive threat by the complementary coalition $N \setminus S$. Classical market games for economies with private goods are examples of cooperative games.

Now then, do the players in $N \setminus S$ have no influence on the worth of S if the coalition S is formed? Obviously, the possibility of some elements of $N \setminus S$ to operate against the actions of the members of the coalition S leads to insufficiency of the classical model. For instance, we consider a group of agents who develop an economic activity which is causing them certain profits (or costs). In a external or internal way a modification (sale, buying, etc.) of this activity is proposed to them. This action will suppose a greater profit to them in case they all agree with the change proposed about the actual situation.

These situations may be interpreted in the following manner. We consider pairs (S, T), with $S, T \subseteq N$ and $S \cap T = \emptyset$. Thus, (S, T) yields a partition of the set N of all players in three groups. Players in S are defenders of modifying the actual situation and they want

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to accept a proposal; players in T do not agree with modifying the situation and they will take action against any change. Finally, the members of $N \setminus (S \cup T)$ are not convinced of the profits derived from the proposal, but they do not think of objecting and stop the action managed by the elements of coalition S.

Thus, in our model we consider the set of all ordered pairs of disjoint coalitions $3^N = \{(S,T) : S, T \subseteq N, S \cap T = \emptyset\}$, and define a function $b : 3^N \to \mathbb{R}$. For each $(S,T) \in 3^N$, the worth $b(S,T)$ represents the maximal gain (whenever $b(S,T) > 0$) or minimal loss (whenever $b(S,T) < 0$) that is obtained when S is the set of players for a change in the activity, the players of T are against the change and the players of $N \setminus (S \cup T)$ are not taking part because they are indifferent. Thus, $b(\emptyset, N)$ indicates the cost (or expense) that is obtained when all players decide to follow with the initial situation and $b(N, \emptyset)$ is the maximal gain that is obtained if all players want a change in the activity. As a consequence, the total profit of the bicooperative game (N, b) is given by $b(N, \emptyset) - b(\emptyset, N)$. This leads us in a natural way into the concept of bicooperative game introduced by Bilbao (2000).

Definition 1 A bicooperative game is a pair (N, b) with N a finite set and b a function $b : 3^N \to \mathbb{R}$ with $b(\emptyset, \emptyset) = 0$.

Similarly to the cooperative case in which each coalition $S \in 2^N$ can be identified with a $\{0,1\}$-vector, each signed coalition $(S,T) \in 3^N$ can be identified with the $\{-1, 0, 1\}$-vector $\mathbf{1}_{(S,T)}$ defined, for all $i \in N$, by

$$\mathbf{1}_{(S,T)}(i) = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

A special kind of bicooperative games has been studied by Felsenthal and Machover (1997) who consider *ternary voting games*. This concept is a generalization of voting games which recognizes abstention as an option alongside *yes* and *no* votes. These games are given by mappings $u : 3^N \to \{-1, 1\}$ satisfying the following three conditions: $u(N, \emptyset) = 1$, $u(\emptyset, N) = -1$, and $\mathbf{1}_{(S,T)}(i) \leq \mathbf{1}_{(S',T')}(i)$ for all $i \in N$, implies $u(S,T) \leq u(S',T')$. A negative outcome, -1, is interpreted as defeat and a positive outcome, 1, as passage of a bill.

More recently, several works by Freixas (2005a, 2005b) and Freixas and Zwicker (2003) have been devoted to the study of voting systems with several ordered levels of approval in the input and in the output. In their model, the abstention is a level of input approval intermediate between *yes* and *no* votes.

A one-point solution concept for cooperative games is a function which assigns to every cooperative game a n-dimensional real vector which represents a payoff distribution over the players. The study of solution concepts is central in cooperative game theory. The most important solution concept is the *Shapley value* as proposed by Shapley (1953). The Shapley value assumes that every player is equally likely to join to any coalition of the same size and all coalitions with the same size are equally likely. The Shapley value $\Phi(v) \in \mathbb{R}^n$ of game v is a weighted average of the marginal contributions of the players and for player $i \in N$, it is given by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-1-s)!}{n!} [v(S \cup \{i\}) - v(S)],$$

where $s = |S|$ and $n = |N|$.

Another form to introduce the Shapley value is based on the marginal worth vectors and corresponds to the following interpretation. Suppose the players enter a room one by one in a randomly chosen order. Each player gets the amount that he contributes to the coalition S already formed into the room when the player i enters the room; that is, i gets $v(S \cup \{i\}) - v(S)$. The Shapley value $\Phi(v)$ distributes to each player $i \in N$, the expected amount that he gets by this procedure, that is,

$$\Phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi_n} [v(\pi^i \cup \{i\}) - v(\pi^i)],$$

where Π_n is the set of all permutations of N and π^i is the set of the predecessors of player i in the order π.

Let us outline the contents of our work. In the next section, we study some properties and characteristics of the lattice 3^N. The aim of the third section is to introduce the Shapley value for a bicooperative game. We obtain an axiomatization of the Shapley value in this context as well as a nice formula to compute it. This value is the only one that satisfies our five axioms. Four of them are extensions of the classical axioms for the Shapley value: linearity, anonymity, dummy and efficiency. The fifth axiom is referred to the structure of the family of signed coalitions. Throughout this paper, we will write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$ respectively.

2 The lattice 3^N

Let $N = \{1, \ldots, n\}$ be a finite set and let $3^N = \{(A, B)\colon\ A, B \subseteq N, A \cap B = \emptyset\}$. Grabisch and Labreuche (2002) proposed a relation in 3^N given by

$$(A, B) \sqsubseteq (C, D) \Longleftrightarrow A \subseteq C, B \supseteq D.$$

The set $(3^N, \sqsubseteq)$ is a partially ordered set (or poset) with the following properties:

1. $(\emptyset, N)$ is the first element: $(\emptyset, N) \sqsubseteq (A, B)$ for all $(A, B) \in 3^N$.
2. $(N, \emptyset)$ is the last element: $(A, B) \sqsubseteq (N, \emptyset)$ for all $(A, B) \in 3^N$.
3. Every pair of elements of 3^N has a join

$$(A, B) \vee (C, D) = (A \cup C, B \cap D)$$

and a meet

$$(A, B) \wedge (C, D) = (A \cap C, B \cup D).$$

Moreover, $(3^N, \sqsubseteq)$ is a finite distributive lattice. Two pairs (A, B) and (C, D) are comparable if $(A, B) \sqsubseteq (C, D)$ or $(C, D) \sqsubseteq (A, B)$; otherwise, (A, B) and (C, D) are incomparable. A chain of 3^N is an induced subposet of 3^N in which any two elements are comparable. In $(3^N, \sqsubseteq)$, all maximal chains have the same number of elements and this number is $2n + 1$. Thus, we can consider the rank function

$$\rho : 3^N \to \{0, 1, \ldots, 2n\}$$

such that $\rho[(\emptyset, N)] = 0$ and $\rho[(S, T)] = \rho[(A, B)] + 1$ if (S, T) covers (A, B), that is, if $(A, B) \sqsubset (S, T)$ and there exists no $(H, J) \in 3^N$ such that $(A, B) \sqsubset (H, J) \sqsubset (S, T)$.

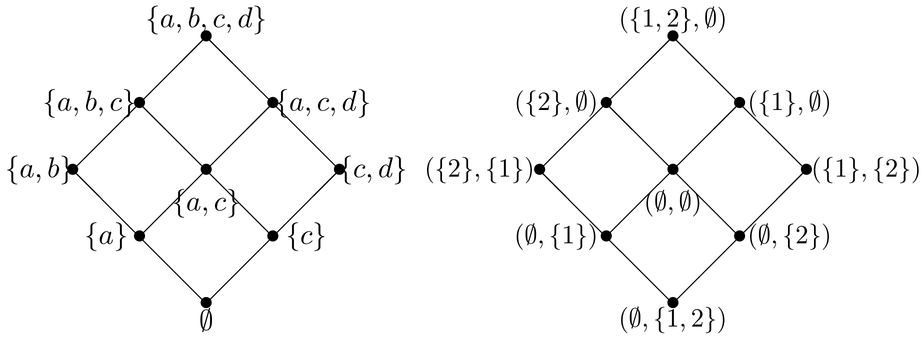

Fig. 1 Example of isomorphism

For the distributive lattice 3^N, let P denote the set of all nonzero $\vee$-irreducible elements. Then P is the disjoint union $C_1 + C_2 + \cdots + C_n$ of the chains

$$C_i = \{(\emptyset, N \setminus i), (i, N \setminus i)\}, \quad 1 \le i \le n = |N|.$$

An order ideal of P is a subset I of P such that if $x \in I$ and $y \le x$, then $y \in I$. The set of all order ideals of P, ordered by inclusion, is the distributive lattice $J(P)$, where the lattice operations $\vee$ and $\wedge$ are just ordinary union and intersection. The fundamental theorem for finite distributive lattices (see Stanley 1986, Theorem 3.4.1) states that the map $\varphi : 3^N \to J(P)$ given by $(A, B) \mapsto \{(X, Y) \in P : (X, Y) \sqsubseteq (A, B)\}$ is an isomorphism (see Fig. 1).

Example Let $N = \{1, 2\}$. Then $P = \{(\emptyset, \{1\}), (\emptyset, \{2\}), (\{2\}, \{1\}), (\{1\}, \{2\})\}$ is the disjoint union of the chains $(\emptyset, \{1\}) \sqsubset (\{2\}, \{1\})$ and $(\emptyset, \{2\}) \sqsubset (\{1\}, \{2\})$. We will denote $a = (\emptyset, \{1\}), b = (\{2\}, \{1\}), c = (\emptyset, \{2\}), d = (\{1\}, \{2\})\}$, and hence

$$J(P) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}.$$

In the following, we will denote by $c(3^N)$ the number of maximal chains in 3^N and by $c([(A, B), (C, D)])$ the number of maximal chains in the sublattice $[(A, B), (C, D)]$.

Proposition 1 *The number of maximal chains of* 3^N *is* $(2n)!/2^n$, *where* $n = |N|$.

Proof The number of maximal chains of 3^N is equal to the number of maximal chains of $J(P)$ and this number is also equal to the number of extensions $e(P)$ of P to a total order (see Stanley 1986, Sect. 3.5).

Since $P = C_1 + \cdots + C_n$, where the chain C_i satisfies $|C_i| = 2$ for $1 \le i \le n$, we can apply the enumeration of lattice paths method from (Stanley 1986, Example 3.5.4), and obtain

$$c(3^N) = e(P) = \binom{2n}{2, \ldots, 2} = \frac{(2n)!}{2^n}. \qquad \square$$

Proposition 2 *For all* $(A, B) \in 3^N$, *the number of maximal chains of the sublattice* $[(\emptyset, N), (A, B)]$ *is* $(n + a - b)!/2^a$, *where* $a = |A|$ *and* $b = |B|$.

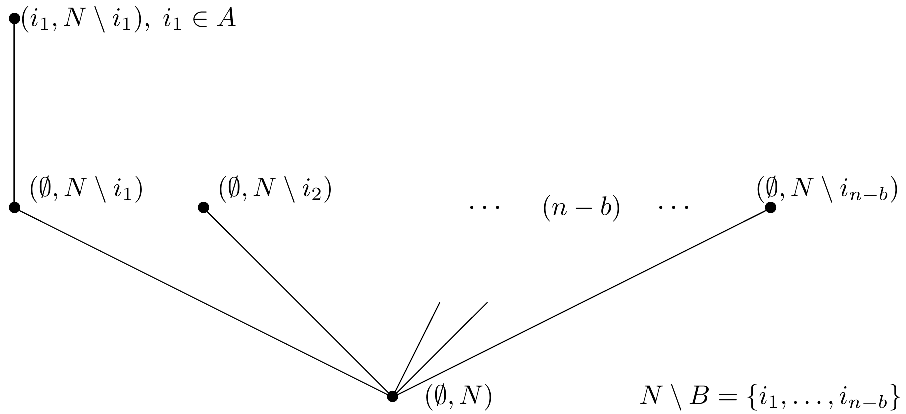

Fig. 2 Chains of the sublattice

Proof It is evident that in the sublattice $[(\emptyset, N), (A, B)]$, there are $n-b$ elements $(\emptyset, N\setminus i)$ with $i\notin B$ (see Fig. 2).

Since $A\subseteq N\setminus B$, then $a\le n-b$ and thus, the set of the irreducible elements of the sublattice can be written as

$$P_{[(\emptyset,N),(A,B)]}=C_1+\cdots+C_a+C_{a+1}+\cdots+C_{a+(n-b-a)}$$

where for all $i_j\in A,\ 1\le j\le a$ and $i_{a+k}\notin A\cup B,\ 1\le k\le n-b-a$, we obtain

$$C_j=\{(\emptyset, N\setminus i_j), (i_j, N\setminus i_j)\},$$
$$C_{a+k}=\{(\emptyset, N\setminus i_{a+k})\}.$$

That is, there are a chains such that $|C_j|=2$ and there are $n-b-a$ chains such that $|C_{a+k}|=1$. Since

$$|C_1|+\cdots+|C_a|+|C_{a+1}|+\cdots+|C_{a+(n-b-a)}|=2a+(n-b-a),$$

we can apply the enumeration of lattice paths method from (Stanley 1986, Sect. 3.5) and we obtain

$$c([(\emptyset, N), (A, B)])=\binom{2a+(n-b-a)}{2,\dots,2,1,\dots,1}=\frac{(n+a-b)!}{2^a}.$$ □

Proposition 3 *Let* $(A, B), (C, D)\in 3^N$ *with* $(A, B)\sqsubseteq (C, D)$. *The number of maximal chains of the sublattice* $[(A, B), (C, D)]$ *is equal to the number of maximal chains of the sublattice* $[(D, C), (B, A)]$.

Proof First of all, note that if $(A, B)\sqsubseteq (C, D)$, then $A\subseteq C$, $B\supseteq D$ and hence $(D, C)\sqsubseteq (B, A)$. Therefore, $[(D, C), (B, A)]$ is a sublattice of 3^N.

Let $\varphi: (3^N, \sqsubseteq)\to(3^N, \sqsubseteq)$ be the map defined by $\varphi(A, B)=(B, A)$. This map is one to one since

$$\varphi(A, B)=\varphi(C, D)\Longleftrightarrow (B, A)=(D, C)\Longleftrightarrow B=D,\ A=C\Longleftrightarrow (A, B)=(C, D).$$

Clearly, if $(A,B) \sqsubset (A_1,B_1) \sqsubset \cdots \sqsubset (A_k,B_k) \sqsubset (C,D)$ is a maximal chain in the sublattice $[(A,B),(C,D)]$ then

$$(D,C) \sqsubset (B_k,A_k) \sqsubset \cdots \sqsubset (B_1,A_1) \sqsubset (B,A)$$

is a maximal chain in the sublattice $[(D,C),(B,A)]$. Finally, it follows that

$$(X,Y) \in [(A,B),(C,D)] \Longleftrightarrow (Y,X) \in [(D,C),(B,A)]. \qquad \square$$

3 The Shapley value for bicooperative games

We denote by $\mathcal{BG}^N$ the real vector space of all bicooperative games on N, that is

$$\mathcal{BG}^N = \{b : 3^N \to \mathbb{R}, b(\emptyset,\emptyset) = 0\}.$$

We consider the *identity games* $\{\delta_{(S,T)} : (S,T) \in 3^N, (S,T) \neq (\emptyset,\emptyset)\}$, the *superior unanimity games* $\{\overline{u}_{(S,T)} : (S,T) \in 3^N, (S,T) \neq (\emptyset,\emptyset)\}$ and the *inferior unanimity games* $\{\underline{u}_{(S,T)} : (S,T) \in 3^N, (S,T) \neq (\emptyset,\emptyset)\}$, which are defined, for any $(S,T) \in 3^N$ such that $(S,T) \neq (\emptyset,\emptyset)$ as follows.

The identity game $\delta_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\delta_{(S,T)}(A,B) = \begin{cases} 1 & \text{if } (A,B) = (S,T), \\ 0 & \text{otherwise.} \end{cases}$$

The superior unanimity game $\overline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is given by

$$\overline{u}_{(S,T)}(A,B) = \begin{cases} 1 & \text{if } (S,T) \sqsubseteq (A,B),\ (A,B) \neq (\emptyset,\emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

The inferior unanimity game $\underline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\underline{u}_{(S,T)}(A,B) = \begin{cases} -1 & \text{if } (A,B) \sqsubseteq (S,T),\ (A,B) \neq (\emptyset,\emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to prove (see Bilbao et al. 2006) that all the above collections are bases of $\mathcal{BG}^N$.

A *value* on $\mathcal{BG}^N$ is a function $\Phi : \mathcal{BG}^N \to \mathbb{R}^n$, which associates to each bicooperative game b a vector $(\Phi_1(b), \ldots, \Phi_n(b))$ which represents the 'a priori' value that every player has in the game b. In order to define a reasonable value for a bicooperative game and following the same issue and interpretation of the Shapley value in the cooperative case, we consider that a player i estimates his participation in game b, evaluating his marginal contributions $b(S \cup i, T) - b(S,T)$ in those signed coalitions $(S \cup i, T)$ that are formed from others (S,T) when i is incorporated to S and his marginal contributions $b(S,T) - b(S, T \cup i)$ in those (S,T) that are formed when i leaves the coalition $T \cup i$.

Thus, a value for player i can be written as

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} [\overline{p}^i_{(S,T)}(b(S \cup i, T) - b(S,T)) + \underline{p}^i_{(S,T)}(b(S,T) - b(S, T \cup i))],$$

where for every (S,T), the coefficient $\overline{p}^i_{(S,T)}$ can be interpreted as the subjective probability that the player i has of joining the coalition S and $\underline{p}^i_{(S,T)}$ as the subjective probability that

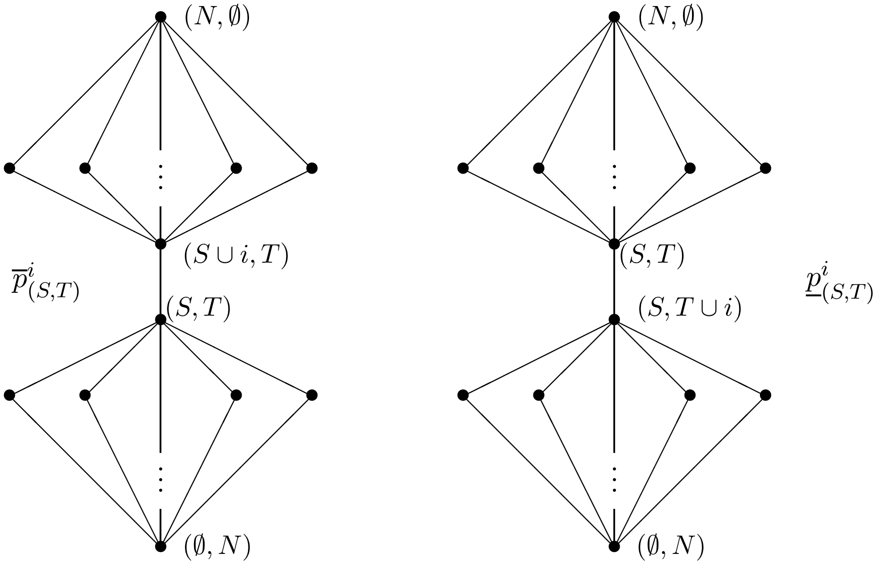

Fig. 3 Chains of the lattice

the player i has of leaving the coalition $T\cup i$. Thus, $\Phi_i(b)$ is the value that the player i can expect in the game b (see Bilbao et al. 2006).

Figure 3 shows the different sequential orders corresponding to the different chains from $(\emptyset,N)$ to $(N,\emptyset)$ which contain (S,T) and $(S\cup i,T)$ and all chains that contain the signed coalitions $(S,T\cup i)$ and (S,T).

If we assume that all sequential orders or chains have the same probability, we can deduce formulas for these probabilities $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ in terms of the number of chains which contain to these coalitions. Applying Propositions 1, 2 and 3, we obtain

$$\begin{aligned}\overline{p}^i_{(S,T)} &= \frac{c([(\emptyset,N),(S,T)])c([(S\cup i,T),(N,\emptyset)])}{c(3^N)}\\ &= \frac{\frac{(n+s-t)!}{2^s}\cdot\frac{(n+t-s-1)!}{2^t}}{\frac{(2n)!}{2^n}}\\ &= \frac{(n+s-t)!(n+t-s-1)!}{(2n)!}2^{n-s-t},\end{aligned}$$

$$\begin{aligned}\underline{p}^i_{(S,T)} &= \frac{c([(\emptyset,N),(S,T\cup i)])\;c([(S,T)(N,\emptyset)])}{c(3^N)}\\ &= \frac{\frac{(n+t-s)!}{2^t}\cdot\frac{(n+s-t-1)!}{2^s}}{\frac{(2n)!}{2^n}}\\ &= \frac{(n+t-s)!(n+s-t-1)!}{(2n)!}2^{n-s-t}.\end{aligned}$$

Taking into account that $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ are independent of player i, and only depend of $s=|S|$ and $t=|T|$, we can establish the following definition.

Definition 2 The Shapley value for the bicooperative game $b \in \mathcal{BG}^N$ is a vector $\Phi \in \mathbb{R}^N$ defined, for each $i \in N$, by

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} [\overline{p}_{s,t}(b(S\cup i,T) - b(S,T)) + \underline{p}_{s,t}(b(S,T) - b(S,T\cup i))]$$

where, for all $(S,T) \in 3^{N\setminus i}$,

$$\overline{p}_{s,t} = \frac{(n+s-t)!(n+t-s-1)!}{(2n)!} 2^{n-s-t}, \qquad \underline{p}_{s,t} = \frac{(n+t-s)!(n+s-t-1)!}{(2n)!} 2^{n-s-t}.$$

Note that $\overline{p}_{s,t} = \underline{p}_{t,s}$.

With the aim to characterize the Shapley value for bicooperative games, we consider a set of reasonable axioms and we prove that the Shapley value is the unique value on $\mathcal{BG}^N$ which satisfies these axioms.

Linearity axiom *For all* $\alpha, \beta \in \mathbb{R}$, *and* $b, w \in \mathcal{BG}^N$,

$$\Phi_i(\alpha b + \beta w) = \alpha \Phi_i(b) + \beta \Phi_i(w), \qquad \textit{for all } i \in N.$$

We now introduce the dummy axiom, understanding that a player is a *dummy player* when his contributions to signed coalitions $(S \cup i, T)$ formed with his incorporation to S and his contributions to signed coalitions (S,T) formed with his desertion of $T \cup i$ coincide exactly with his individual contributions, that is, a player $i \in N$ is a dummy in $b \in \mathcal{BG}^N$ if, for every $(S,T) \in 3^{N\setminus i}$, it holds

$$b(S\cup i,T) - b(S,T) = b(\{i\},\emptyset),$$
$$b(S,T) - b(S,T\cup i) = -b(\emptyset,\{i\}).$$

Note that if $i \in N$ is a dummy in $b \in \mathcal{BG}^N$ then, for all $(S,T) \in 3^{N\setminus i}$,

$$b(S\cup i,T) - b(S,T\cup i) = b(\{i\},\emptyset) - b(\emptyset,\{i\}).$$

Since a dummy player i in a game b has no meaningful strategic role in the game, the value that this player should expect in the game b must exactly be the sum up of his contributions.

Dummy axiom *If player* $i \in N$ *is dummy in* $b \in \mathcal{BG}^N$, *then*

$$\Phi_i(b) = b(\{i\},\emptyset) - b(\emptyset,\{i\}).$$

In a similar way to the cooperative case, for the comparison of roles in a game to be meaningful, the evaluation of a particular position should depend on the structure of the game but not on the labels of the players.

Anonymity axiom *For all* $b \in \mathcal{BG}^N$ *and for any permutation* π *over* N, *it holds that* $\Phi_{\pi i}(\pi b) = \Phi_i(b)$ *for all* $i \in N$, *where* $\pi b(\pi S, \pi T) = b(S,T)$ *and* $\pi S = \{\pi i : i \in S\}$.

In a cooperative game, it is assumed that all players decide to cooperate among them and form the grand coalition N. This leads to the problem of distributing the amount $v(N)$ among them, that is, $v(N) - v(\emptyset)$. This amount represents the profit that is obtained if all players pass away the initial situation, represented by the coalition, $S = \emptyset$, and the whole group of players decide to form the coalition N. Taking into account that, in a bicooperative game, $b(\emptyset, N)$ is the cost (or expense) that is obtained when all players decide to follow with the initial situation and $b(N, \emptyset)$ is the maximal gain that is obtained if all players want a change in the activity, then the net profit is given by $b(N, \emptyset) - b(\emptyset, N)$. From this perspective, the value Φ must satisfy the following axiom.

Efficiency axiom *For every $b \in \mathcal{BG}^N$, it holds*

$$\sum_{i \in N} \Phi_i(b) = b(N, \emptyset) - b(\emptyset, N).$$

It is easy to check that our Shapley value for bicooperative games verifies the above axioms. But this value is not the unique value which satisfies these four axioms. For instance, the value $\Phi(b)$ defined, for $b \in \mathcal{BG}^N$ and $i \in N$, by

$$\Phi_i(b) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} [b(S \cup i, N \setminus (S \cup i)) - b(S, N \setminus S)],$$

also verifies these axioms. However, note that, for any bicooperative game $b \in \mathcal{BG}^N$, this value is the Shapley value corresponding to the cooperative game (N, v), where $v : 2^N \to \mathbb{R}$ is defined by $v(A) = b(A, N \setminus A)$ if $A \neq \emptyset$, and $v(\emptyset) = 0$. This value is not satisfactory for any bicooperative game in the sense that it only consider the contributions to signed coalitions in which all players take part. Moreover, there is an infinity of different bicooperative games which give rise to the same cooperative game.

For these reasons, if we want to obtain an axiomatic characterization of our Shapley value for bicooperative games, we need to introduce an additional axiom. Previously, we show that a value on $\mathcal{BG}^N$ that satisfies the above four axioms is given by the expression

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} [\overline{p}^i_{(S,T)}(b(S \cup i, T) - b(S, T)) + \underline{p}^i_{(S,T)}(b(S, T) - b(S, T \cup i))],$$

where $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ satisfy some conditions. We prove this result in several steps. First of all, we show that a value for player i satisfying the linearity and dummy axioms can be expressed as a linear combination of his contributions.

Theorem 4 *Let Φ_i be a value for player $i \in N$ which satisfies the linearity and dummy axioms. Then, for every $b \in \mathcal{BG}^N$,*

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} [\overline{p}^i_{(S,T)}(b(S \cup i, T) - b(S, T)) + \underline{p}^i_{(S,T)}(b(S, T) - b(S, T \cup i))]$$

where

$$\sum_{(S,T) \in 3^{N \setminus i}} \overline{p}^i_{(S,T)} = 1, \quad \textit{and} \quad \sum_{(S,T) \in 3^{N \setminus i}} \underline{p}^i_{(S,T)} = 1.$$

Proof The set of identity games is a basis of $\mathcal{BG}^N$, and each game $b \in \mathcal{BG}^N$ can be written as

$$b = \sum_{\{(S,T)\in 3^N:(S,T)\neq(\emptyset,\emptyset)\}} b(S,T)\delta_{(S,T)}.$$

By the linearity axiom,

$$\Phi_i(b) = \sum_{\{(S,T)\in 3^N:(S,T)\neq(\emptyset,\emptyset)\}} \Phi_i(\delta_{(S,T)})b(S,T).$$

We denote by $a^i_{(S,T)} = \Phi_i(\delta_{(S,T)})$ for all $(S,T) \neq (\emptyset,\emptyset)$ and thus, the value $\Phi_i(b)$ is given by

$$\begin{aligned}
&\sum_{(S,T)\in 3^N} a^i_{(S,T)} b(S,T) \\
&= \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S,T)} b(S,T) + \sum_{\{(S,T)\in 3^N:i\in S\}} a^i_{(S,T)} b(S,T) + \sum_{\{(S,T)\in 3^N:i\in T\}} a^i_{(S,T)} b(S,T) \\
&= \sum_{\{(S,T)\in 3^{N\setminus i}:(S,T)\neq(\emptyset,\emptyset)\}} a^i_{(S,T)} b(S,T) + \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S\cup i,T)} b(S\cup i,T) \\
&\quad + \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S,T\cup i)} b(S,T\cup i) \\
&= \sum_{\{(S,T)\in 3^{N\setminus i}:(S,T)\neq(\emptyset,\emptyset)\}} (a^i_{(S,T)} b(S,T) + a^i_{(S\cup i,T)} b(S\cup i,T) + a^i_{(S,T\cup i)} b(S,T\cup i)) \\
&\quad + a^i_{(\{i\},\emptyset)} b(\{i\},\emptyset) + a^i_{(\emptyset,\{i\})} b(\emptyset,\{i\}).
\end{aligned}$$

Let us consider the games $w^i_{(A,B)} : 3^N \to \mathbb{R}$ where, for each $(A,B) \in 3^{N\setminus i}$, the game $w^i_{(A,B)}$ is defined by

$$w^i_{(A,B)}(S,T) = \begin{cases} w^i_{(A,B)}(S\setminus i,T) & \text{if } i\in S, \\ w^i_{(A,B)}(S,T\setminus i) & \text{if } i\in T, \\ 1 & \text{if } i\notin S\cup T, (\emptyset,\emptyset)\neq(S,T)\sqsubseteq(A,B), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, player i is a dummy in $w^i_{(A,B)}$ for each $(A,B) \in 3^{N\setminus i}$ and hence $\Phi_i(w^i_{(A,B)}) = 0$ by the dummy axiom. If we apply the above equality to the game $w^i_{(A,B)}$ we get

$$\sum_{\{(S,T)\in 3^{N\setminus i}:(\emptyset,\emptyset)\neq(S,T)\sqsubseteq(A,B)\}} (a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)}) = 0.$$

We show, by induction on $\rho[(S,T)]$, the rank of the signed coalitions, that for all $(S,T) \in 3^{N\setminus i}$, $(S,T) \neq (\emptyset,\emptyset)$, it holds that $a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} = 0$. Note that the first element in $(3^{N\setminus i}, \sqsubseteq)$ is $(\emptyset, N\setminus i)$, and so $\rho[(\emptyset, N\setminus i)] = 0$. Hence

$$\sum_{\{(S,T)\in 3^{N\setminus i}:(S,T)\sqsubseteq(\emptyset,N\setminus i)\}} (a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)}) = a^i_{(\emptyset,N\setminus i)} + a^i_{(\{i\},N\setminus i)} + a^i_{(\emptyset,N)} = 0.$$

Now assume the property for $(H, J) \in 3^{N\setminus i}$ with $\rho[(H, J)] \leq k - 1$ and suppose that $(S, T) \in 3^{N\setminus i}$ has $\rho[(S, T)] = k$. Then

$$\begin{aligned}
\Phi_i(w^i_{(S,T)}) &= \sum_{\{(H,J)\in 3^{N\setminus i}:(\emptyset,\emptyset)\neq(H,J)\sqsubseteq(S,T)\}} (a^i_{(H,J)} + a^i_{(H\cup i,J)} + a^i_{(H,J\cup i)}) \\
&= a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} \\
&\quad + \sum_{\{(H,J)\in 3^{N\setminus i}:(\emptyset,\emptyset)\neq(H,J)\sqsubset(S,T)\}} (a^i_{(H,J)} + a^i_{(H\cup i,J)} + a^i_{(H,J\cup i)}) \\
&= a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} = 0,
\end{aligned}$$

where the last but one equality follows from the induction hypothesis, and the last one follows from the dummy axiom. Now for each $(S, T) \in 3^{N\setminus i}$, define

$$\overline{p}^i_{(\emptyset,\emptyset)} = a^i_{(\{i\},\emptyset)}, \qquad \underline{p}^i_{(\emptyset,\emptyset)} = -a^i_{(\emptyset,\{i\})}, \qquad \overline{p}^i_{(S,T)} = a^i_{(S\cup i,T)}, \qquad \underline{p}^i_{(S,T)} = -a^i_{(S,T\cup i)}$$

and we compute

$$\begin{aligned}
\Phi_i(b) &= \sum_{(S,T)\in 3^{N\setminus i}} [(\underline{p}^i_{(S,T)} - \overline{p}^i_{(S,T)})b(S,T) + \overline{p}^i_{(S,T)}b(S\cup i,T) - \underline{p}^i_{(S,T)}b(S,T\cup i)] \\
&= \sum_{(S,T)\in 3^{N\setminus i}} [\overline{p}^i_{(S,T)}(b(S\cup i,T) - b(S,T)) + \underline{p}^i_{(S,T)}(b(S,T) - b(S,T\cup i))].
\end{aligned}$$

Finally, it is easy to check that player i is a dummy in the games $\overline{u}_{(\{i\},N\setminus i)}$ and $\underline{u}_{(N\setminus i,\{i\})}$, and hence

$$\begin{aligned}
\sum_{(S,T)\in 3^{N\setminus i}} \overline{p}^i_{(S,T)} &= \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S\cup i,T)} = \sum_{\{(S,T)\in 3^N:i\in S\}} a^i_{(S,T)} \\
&= \sum_{\{(S,T)\in 3^N:i\in S\}} \Phi_i(\delta_{(S,T)}) = \Phi_i\left(\sum_{\{(S,T)\in 3^N:i\in S\}} \delta_{(S,T)}\right) \\
&= \Phi_i(\overline{u}_{(\{i\},N\setminus i)}) = \overline{u}_{(\{i\},N\setminus i)}(\{i\},\emptyset) - \overline{u}_{(\{i\},N\setminus i)}(\emptyset,\{i\}) = 1.
\end{aligned}$$

$$\begin{aligned}
\sum_{(S,T)\in 3^{N\setminus i}} \underline{p}^i_{(S,T)} &= \sum_{(S,T)\in 3^{N\setminus i}} -a^i_{(S,T\cup i)} = \sum_{\{(S,T)\in 3^N:i\in T\}} -a^i_{(S,T)} \\
&= \sum_{\{(S,T)\in 3^N:i\in T\}} -\Phi_i(\delta_{(S,T)}) = \Phi_i\left(\sum_{\{(S,T)\in 3^N:i\in T\}} -\delta_{(S,T)}\right) \\
&= \Phi_i(\underline{u}_{(N\setminus i,\{i\})}) = \underline{u}_{(N\setminus i,\{i\})}(\{i\},\emptyset) - \underline{u}_{(N\setminus i,\{i\})}(\emptyset,\{i\}) = 1. \qquad \square
\end{aligned}$$

Now, we show that if we add the anonymity axiom to the linearity and dummy axioms, the coefficients $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ only depend of the cardinality of S and T.

Theorem 5 *Let Φ_i be a value for player $i \in N$ defined, for every game $b \in \mathcal{BG}^N$, by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} [\overline{p}^i_{(S,T)}(b(S\cup i,T) - b(S,T)) + \underline{p}^i_{(S,T)}(b(S,T) - b(S,T\cup i))].$$

If Φ_i *satisfies the anonymity axiom, then* $\overline{p}^i_{(S,T)} = \overline{p}_{s,t}$ *and* $\underline{p}^i_{(S,T)} = \underline{p}_{s,t}$ *for all* $(S,T) \in 3^{N\setminus i}$ *with* $s = |S|$ *and* $t = |T|$.

Proof Let Φ_i be a value for player i given by

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} [\overline{p}^i_{(S,T)}(b(S\cup i, T) - b(S,T)) + \underline{p}^i_{(S,T)}(b(S,T) - b(S, T\cup i))].$$

Let (S_1, T_1) and (S_2, T_2) be signed coalitions in $3^{N\setminus i}$ such that $(S_1, T_1) \neq (\emptyset, \emptyset) \neq (S_2, T_2)$ satisfying that $|S_1| = |S_2| < n-1$ and $|T_1| = |T_2| < n-1$. Consider a permutation π of N that takes $\pi S_1 = S_2$ and $\pi T_1 = T_2$ while leaving i fixed. Then $\pi\delta_{(S_1,T_1)} = \delta_{(S_2,T_2)}$ and

$$\overline{p}^i_{(S_1,T_1)} = \Phi_i(\delta_{(S_1\cup i, T_1)}) = \Phi_i(\delta_{(S_2\cup i, T_2)}) = \overline{p}^i_{(S_2,T_2)},$$
$$\underline{p}^i_{(S_1,T_1)} = -\Phi_i(\delta_{(S_1, T_1\cup i)}) = -\Phi_i(\delta_{(S_2, T_2\cup i)}) = \underline{p}^i_{(S_2,T_2)},$$

where the second equality follows from the anonymity axiom.

Now, let $i, j \in N, i \neq j$ and let $(S,T) \in 3^{N\setminus\{i,j\}}$. Consider the permutation π of N that interchanges i and j while leaving the remaining players fixed. Then $\pi\delta_{(S,T)} = \delta_{(S,T)}$ and

$$\overline{p}^i_{(S,T)} = \Phi_i(\delta_{(S\cup i, T)}) = \Phi_j(\delta_{(S\cup j, T)}) = \overline{p}^j_{(S,T)},$$
$$\underline{p}^i_{(S,T)} = -\Phi_i(\delta_{(S, T\cup i)}) = -\Phi_j(\delta_{(S, T\cup j)}) = \underline{p}^j_{(S,T)}.$$

Moreover,

$$\overline{p}^i_{(N\setminus i,\emptyset)} = \Phi_i(\delta_{(N,\emptyset)}) = \Phi_j(\delta_{(N,\emptyset)}) = \overline{p}^j_{(N\setminus j,\emptyset)},$$
$$\underline{p}^i_{(\emptyset,N\setminus i)} = -\Phi_i(\delta_{(\emptyset,N)}) = -\Phi_j(\delta_{(\emptyset,N)}) = \underline{p}^j_{(\emptyset,N\setminus j)}.$$

Hence, for every $(S,T) \in 3^{N\setminus i}$ there exist $\overline{p}_{s,t}$ and $\underline{p}_{s,t}$ such that $\overline{p}^i_{(S,T)} = \overline{p}_{s,t}$ and $\underline{p}^i_{(S,T)} = \underline{p}_{s,t}$ for all $i \in N$. □

The following theorem characterizes the values $\Phi = (\Phi_1, \ldots, \Phi_n)$ which satisfy the above axioms and are efficient.

Theorem 6 *Let* $\Phi = (\Phi_1, \ldots, \Phi_n)$ *be a value on* $\mathcal{BG}^N$ *defined, for every game b and for all* $i \in N$, *by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} [\overline{p}_{s,t}(b(S\cup i, T) - b(S,T)) + \underline{p}_{s,t}(b(S,T) - b(S, T\cup i))].$$

Then, the value Φ *satisfies the efficiency axiom if and only if it is satisfied*

$$\overline{p}_{n-1,0} = \frac{1}{n}, \qquad \underline{p}_{0,n-1} = \frac{1}{n},$$

and

$$(n-s-t)\overline{p}_{s,t} + t\underline{p}_{s,t-1} = (n-s-t)\underline{p}_{s,t} + s\overline{p}_{s-1,t}$$

for all $0 \le s, t \le n-1$ *and* $0 < s+t \le n-1$.

Proof For every $b \in \mathcal{BG}^N$ we have that $\sum_{i\in N} \Phi_i(b)$ is equal to

$$\begin{aligned}
&\sum_{i\in N} \sum_{(S,T)\in 3^{N\setminus i}} [\overline{p}_{s,t}(b(S\cup i, T) - b(S,T)) + \underline{p}_{s,t}(b(S,T) - b(S, T\cup i))] \\
&= \sum_{i\in N} \sum_{(S,T)\in 3^{N\setminus i}} [\overline{p}_{s,t} b(S\cup i, T) - \underline{p}_{s,t} b(S, T\cup i) + (-\overline{p}_{s,t} + \underline{p}_{s,t}) b(S,T)] \\
&= \sum_{(S,T)\in 3^N} b(S,T)[s\overline{p}_{s-1,t} - t\underline{p}_{s,t-1} + (n-s-t)(-\overline{p}_{s,t} + \underline{p}_{s,t})] \\
&= b(N,\emptyset) n\overline{p}_{n-1,0} - b(\emptyset, N) n\underline{p}_{0,n-1} \\
&\quad + \sum_{\substack{(S,T)\in 3^N \\ (S,T)\notin\{(\emptyset,\emptyset),(\emptyset,N),(N,\emptyset)\}}} b(S,T)[s\overline{p}_{s-1,t} - t\underline{p}_{s,t-1} + (n-s-t)(-\overline{p}_{s,t} + \underline{p}_{s,t})].
\end{aligned}$$

If the coefficients satisfy the relations for the coefficients, then Φ satisfies the efficiency axiom.

Conversely, fix $(S,T) \in 3^N$, $(S,T) \neq (\emptyset, \emptyset)$, and applying the preceding equality to the identity game $\delta_{(S,T)}$, we have

$$\sum_{i\in N} \Phi_i(\delta_{(S,T)}) = \begin{cases} n\overline{p}_{n-1,0} & \text{if } (S,T) = (N,\emptyset), \\ -n\underline{p}_{0,n-1} & \text{if } (S,T) = (\emptyset, N), \\ s\overline{p}_{s-1,t} - t\underline{p}_{s,t-1} + (n-s-t)(\underline{p}_{s,t} - \overline{p}_{s,t}) & \text{otherwise.} \end{cases}$$

Thus, if Φ satisfies the efficiency axiom, the relations for the coefficients are true. □

As we have already indicated, these four axioms are not sufficient to characterize the Shapley value for bicooperative games. Now, we introduce an additional axiom and prove that our Shapley value is the unique value on $\mathcal{BG}^N$ that verifies the five axioms. This new axiom will take into account the structure of the set of the signed coalitions.

First of all, note that the signed coalitions $(S \setminus j, T)$ and $(S, T \cup i)$ where $j \in S$ and $i \notin S \cup T$ have the same rank $\rho[(S \setminus j, T)] = \rho[(S, T \cup i)] = n + s - t - 1$. However, the number of maximal chains in the sublattice $[(\emptyset, N), (S \setminus j, T)]$ is not the same than the number of maximal chains in $[(\emptyset, N), (S, T \cup i)]$ since, by Proposition 2,

$$c([(\emptyset, N), (S \setminus j, T)]) = \frac{(n+s-1-t)!}{2^{s-1}},$$
$$c([(\emptyset, N), (S, T \cup i)]) = \frac{(n+s-t-1)!}{2^s}.$$

Hence, beginning from the signed coalition $(\emptyset, N)$, the probability of formation of the signed coalition (S,T) with the incorporation of one player j to $(S \setminus j, T)$ must be distinct to the probability of formation (S,T) with the desertion of one player i in $(S, T \cup i)$.

In analogous form, if we consider $(S, T \setminus k)$ with $k \in T$ and $(S \cup i, T)$ which have the same rank, the number of maximal chains in $[(S, T \setminus k), (N, \emptyset)]$ is not equal to number of maximal chains in $[(S \cup i, T), (N, \emptyset)]$. Therefore the probability of formation of $(N, \emptyset)$ beginning from $(S, T \setminus k)$ when one player k leaves the coalition T must be distinct to the probability of formation of $(N, \emptyset)$ when one player i form the signed coalition $(S \cup i, T)$.

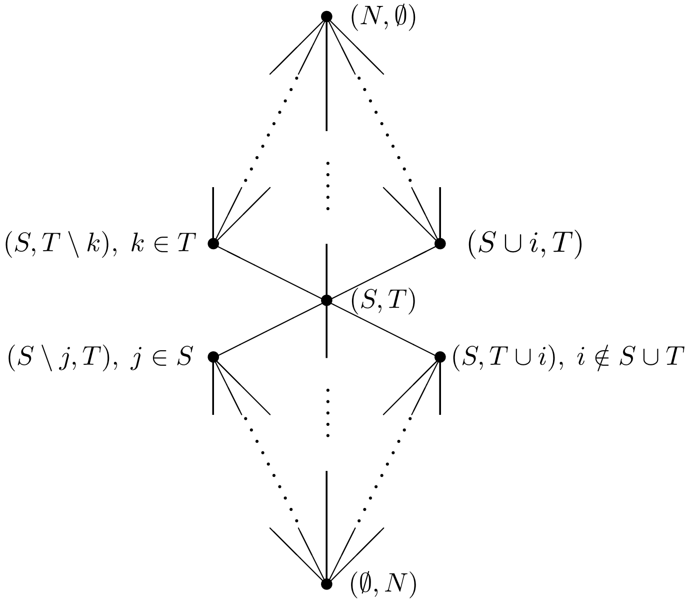

Fig. 4 Chains of the structural axiom

Taking into account these considerations, the values that one player must obtain in the identity games must be proportional to the number of maximal chains in the corresponding sublattices. It must be also considered that one value verifying the above four axioms assigns a non-negative real number to one player i in the identity game $\delta_{(S,T)}$ if this player belongs S and a non-positive real number if the player i belongs T. From this point of view, our value must satisfy the following axiom (see Fig. 4).

Structural axiom *For every* $(S,T)\in 3^{N\setminus i}$, $j\in S$ *and* $k\in T$, *it holds*

$$\frac{c([(\emptyset,N),(S\setminus j,T)])}{c([(\emptyset,N),(S,T\cup i)])}=-\frac{\Phi_j(\delta_{(S,T)})}{\Phi_i(\delta_{(S,T\cup i)})},\qquad \frac{c([(S,T\setminus k),(N,\emptyset)])}{c([(S\cup i,T),(N,\emptyset)])}=-\frac{\Phi_k(\delta_{(S,T)})}{\Phi_i(\delta_{(S\cup i,T)})}.$$

Theorem 7 *Let* Φ *be a value on* $\mathcal{BG}^N$. *The value* Φ *is the Shapley value if and only if* Φ *satisfies the efficiency axiom and each component satisfies linearity, dummy, anonymity and structural axioms.*

Proof If Φ is a value that satisfies linearity, dummy, anonymity and efficiency, then

$$\Phi_i(b)=\sum_{(S,T)\in 3^{N\setminus i}}[\overline{p}_{s,t}(b(S\cup i,T)-b(S,T))+\underline{p}_{s,t}(b(S,T)-b(S,T\cup i))]$$

and the coefficients $\overline{p}_{s,t}$ and $\underline{p}_{s,t}$ satisfy

$$\overline{p}_{n-1,0}=\frac{1}{n},\qquad \underline{p}_{0,n-1}=\frac{1}{n},$$

and

$$(n-s-t)\overline{p}_{s,t} + t\underline{p}_{s,t-1} = (n-s-t)\underline{p}_{s,t} + s\overline{p}_{s-1,t}. \tag{1}$$

Taking into account that the value Φ verifies the structural axiom then

$$\overline{p}_{s-1,t} = 2\underline{p}_{s,t}, \tag{2}$$

$$\underline{p}_{s,t-1} = 2\overline{p}_{s,t}. \tag{3}$$

We prove that these coefficients, verifying all above conditions, are determined in unique form. Indeed, consider a coalition (S,T) with $|S| = n-1$ and $|T| = 0$. If we apply (1) this coalition, we obtain

$$\overline{p}_{n-1,0} = \underline{p}_{n-1,0} + (n-1)\overline{p}_{n-2,0}$$

and by (2), $\overline{p}_{n-2,0} = 2\underline{p}_{n-1,0}$. Taking into account that $\overline{p}_{n-1,0} = \frac{1}{n}$ and combining the above equalities, we have that

$$\frac{1}{n} = (1 + 2(n-1))\underline{p}_{n-1,0}$$

and hence

$$\underline{p}_{n-1,0} = \frac{1}{n(2n-1)} = \frac{1!(2n-2)!}{2^{n-1}(2n)!}2^n, \qquad \overline{p}_{n-2,0} = \frac{2}{n(2n-1)} = \frac{1!(2n-2)!}{2^{n-2}(2n)!}2^n.$$

In similar way, if we apply (1) and (2) to a signed coalition (S,T) with $|S| = n-2$ and $|T| = 0$, we get

$$2\overline{p}_{n-2,0} = 2\underline{p}_{n-2,0} + (n-2)\overline{p}_{n-3,0},$$
$$\overline{p}_{n-3,0} = 2\underline{p}_{n-2,0},$$

and hence

$$\underline{p}_{n-2,0} = \frac{2!(2n-3)!}{2^{n-2}(2n)!}2^n, \qquad \overline{p}_{n-3,0} = \frac{2!(2n-3)!}{2^{n-3}(2n)!}2^n.$$

If we assume that

$$\underline{p}_{s+1,0} = \frac{(n-s-1)!(n+s)!}{2^{s+1}(2n)!}2^n, \qquad \overline{p}_{s,0} = \frac{(n-s-1)!(n+s)!}{2^{s}(2n)!}2^n$$

then, for $|S| = s$ and $|T| = 0$, applying (1) and (2),

$$(n-s)\overline{p}_{s,0} = (n-s)\underline{p}_{s,0} + s\overline{p}_{s-1,0},$$
$$\overline{p}_{s-1,0} = 2\underline{p}_{s,0},$$

and combining both expressions, we obtain, for $1 \le s \le n-1$,

$$\underline{p}_{s,0} = \frac{(n-s)!(n+s-1)!}{2^{s}(2n)!}2^n, \qquad \overline{p}_{s-1,0} = \frac{(n-s)!(n+s-1)!}{2^{s-1}(2n)!}2^n.$$

If we apply the same reasoning with (1) and (3) beginning with a coalition (S,T) with $|S|=0$ and $|T|=n-1$, we obtain, for $1\le t\le n-1$,

$$\overline{p}_{0,t}=\frac{(n-t)!(n+t-1)!}{2^t(2n)!}2^n,\qquad \underline{p}_{0,t-1}=\frac{(n-t)!(n+t-1)!}{2^{t-1}(2n)!}2^n.$$

If we now consider (S,T) with $|S|=s$ and $|T|=1$, we apply (1) and (3),

$$(n-s-1)\overline{p}_{s,1}+\underline{p}_{s,0}=(n-s-1)\underline{p}_{s,1}+s\overline{p}_{s-1,1},$$

$$\overline{p}_{s,1}=\frac{1}{2}\underline{p}_{s,0},\qquad \overline{p}_{s-1,1}=\frac{1}{2}\underline{p}_{s-1,0},$$

and substitute the values already obtained, then

$$\overline{p}_{s-1,1}=\frac{(n-s+1)!(n+s-2)!}{2^s(2n)!}2^n,\qquad \underline{p}_{s,1}=\frac{(n-s+1)!(n+s-2)!}{2^{s+1}(2n)!}2^n.$$

If we assume that

$$\overline{p}_{s-1,t-1}=\frac{(n-s+t-1)!(n+s-t)!}{2^{s+t-2}(2n)!}2^n,\qquad \underline{p}_{s,t-1}=\frac{(n-s+t-1)!(n+s-t)!}{2^{s+t-1}(2n)!}2^n,$$

then applying (3) we obtain, for all $0\le s,t\le n-1$ and $s+t\le n-1$,

$$\overline{p}_{s,t}=\frac{(n+s-t)!(n+t-s-1)!}{2^{s+t}(2n)!}2^n.$$

Finally, by (1) and (2),

$$(n-s-t)\overline{p}_{s,t}+t\underline{p}_{s,t-1}=(n-s-t)\underline{p}_{s,t}+s\overline{p}_{s-1,t},$$
$$\overline{p}_{s-1,t}=2\underline{p}_{s,t},$$

it holds that

$$\underline{p}_{s,t}=\frac{(n+t-s)!(n+s-t-1)!}{2^{s+t}(2n)!}2^n$$

for all $0\le s,t\le n-1$ and $s+t\le n-1$. □

4 Concluding remarks

We were initially motivated by the interest in modeling situations such as described in the introduction, in which every player can choose with three different options. The bicooperative games allow us to model these situations and one of the first questions is how to define solution concepts. Our definition and study about the Shapley value for bicooperative games have been based on the classical interpretation of the Shapley value which is one of the most important solution concepts for cooperative games. The Shapley value in a cooperative game assumes that every player is equally likely to join to any coalition of the same size and all coalitions with the same size are equally probably. Now then, in a bicooperative game, every player i can contribute to one signed coalition (S,T) where $i\notin S\cup T$, in two different ways: he can leave the coalition $T \cup i$ and his contribution is $b(S, T) - b(S, T \cup i)$ or he can be incorporated to the coalition of defenders of the change S and his contribution is $b(S \cup i, T) - b(S, T)$. Thus, our definition of the Shapley value assigns different probability distributions $\{\overline{p}^i_{(S,T)}\}$ and $\{\underline{p}^i_{(S,T)}\}$ to the different marginal contributions.

Another form to introduce the Shapley value is to suppose that the formation of the top coalition $(N, \emptyset)$ is based on a sequential process beginning from the bottom coalition $(\emptyset, N)$, where in each step a different player is incorporated to the first coalition or a different player leaves the second one. In each one of these processes, a player i can evaluate his contribution when he is incorporated to a coalition S (superior marginal worth) or his contribution when he leaves a coalition T (inferior marginal worth). The sum of the inferior marginal worth and the superior marginal worth is the contribution for the player i in that process. If we assume that all sequential processes or maximal chains have the same probability, the Shapley value for the player i can be defined as the quotient between the sum of his marginal worths associated to all maximal chains from $(\emptyset, N)$ until $(N, \emptyset)$, and the total number of maximal chains. In (Bilbao et al. 2006), the identity between the Shapley value introduced in this way and the Shapley value studied in the underlying paper is shown.

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