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The core and the Weber set for bicooperative games

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Abstract This paper studies two classical solution concepts for the structure of bicooperative games. First, we define the core and the Weber set of a bicooperative game and prove that the core is always contained in the Weber set. Next, we introduce a special class of bicooperative games, the so-called bisupermodular games, and show that these games are the only ones in which the core and the Weber set coincide.

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1 Introduction

A cooperative game with transferable utility is given by a finite set of players and a real-valued worth function defined on the set of all the subsets, or *coalitions*, of players such that the worth of the empty set is zero. For each coalition, the worth can be interpreted as the maximal gain or minimal cost that the players in this coalition can achieve by themselves against the best offensive threat by the complementary coalition. Classical market games for economies with private goods are examples of cooperative games. Here, we say that such a game has *orthogonal coalitions* (see Myerson 1991, Chap. 9).

Games with non-orthogonal coalitions are games in which the worth of a coalition depends on the actions of its complementary coalition. Clearly, social

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situations involving externalities and public goods are such cases. For instance, the joint owners of a building are considering hiring a gardener to work in the common areas of their residence. The garden is a public good. Each owner can decide to support the proposal or to veto it. However, some of them may decide not to take part in the decision making and would thus not necessarily be *defenders* or *detractors* of the project.[1]

Situations of this kind may be modeled in the following manner. We consider ordered pairs of disjoint coalitions of players. Each such pair yields a partition of the set of all players in three groups. Players in the first coalition are in favor of the proposal, and players in the second coalition object to it. The remaining players are not convinced of its benefits, but they have no intention of objecting to it. This leads us in a natural way into the concept of *bicooperative game* introduced by Bilbao (2000).

A central question in game theory is to define a solution concept for a game, or a class of games. A solution concept for a class of games is a function which assigns to every game a set of real-valued vectors, each one of them represents a payoff distribution among the players. The *core* (Gillies 1953) is one of the most studied solution concepts. The core of a game consists of all payoff vectors which distribute the worth of the grand coalition under the condition that the players in each coalition receive at least the amount that they can obtain by cooperating. The core is a very natural solution concept, and it is nonempty for convex games (Shapley 1971). However, in many cases, it is empty. This leads us to considering other solution concepts. In 1978, Weber (1988) proposed a set that is always nonempty. It is now called the *Weber set*. Weber showed that the core of any cooperative game is a subset of the Weber set, and Ichiishi (1981) proved that if the Weber set is a subset of the core of a game, then the game is convex.

In this paper, we extend the above solution concepts to bicooperative games. Since the Weber set of a bicooperative game is the convex hull of its marginal worth vectors, it is nonempty. Moreover, we prove that the core is always contained in the Weber set. In establishing this relation, the class of bisupermodular games, defined in the fourth section, play an important role. We show that bisupermodular games are the only ones for which the Weber set and the core coincide, which establishes a characterization of these games.

2 Bicooperative games

Let $N = \{1, \ldots, n\}$ be a finite set and $3^N = \{(A, B) : A, B \subseteq N, \;\; A \cap B = \emptyset\}$. Grabisch and Labreuche (2005a) proposed the partial order in 3^N given by

$$(A, B) \sqsubseteq (C, D) \Longleftrightarrow A \subseteq C, \; B \supseteq D.$$

[1] This is the case with multicriteria decision making when underlying scales are bipolar, i.e., a central value exists on each scale and it is considered a neutral value.

We denote by $\sqsubset$ the relation defined by means of the strict inclusion, that is, $(A,B) \sqsubset (C,D)$ if and only if $A \subset C,\ B \supset D$.

The set $\left(3^N, \sqsubseteq\right)$ is a partially ordered set (poset) with the following properties:

1. $(\emptyset, N)$ is the first element: $(\emptyset, N) \sqsubseteq (A,B)$ for all $(A,B) \in 3^N$.
2. $(N, \emptyset)$ is the last element: $(A,B) \sqsubseteq (N,\emptyset)$ for all $(A,B) \in 3^N$.
3. Every pair $\{(A,B), (C,D)\}$ of elements of 3^N has a join

$$(A,B) \vee (C,D) = (A \cup C, B \cap D),$$

and a meet

$$(A,B) \wedge (C,D) = (A \cap C, B \cup D).$$

Moreover, $\left(3^N, \sqsubseteq\right)$ is a finite distributive lattice.

Two pairs (A,B) and (C,D) in 3^N are *comparable* if $(A,B) \sqsubseteq (C,D)$ or $(C,D) \sqsubseteq (A,B)$; otherwise, (A,B) and (C,D) are *non-comparable*. A *chain* in 3^N is an induced subposet of 3^N in which any two elements are comparable. Moreover, if two pairs are comparable, there exists at least one chain that contains them. In $\left(3^N, \sqsubseteq\right)$, all maximal chains have the same number of elements and this number is $2n+1$.

We model above mentioned class of non-orthogonal situations by mean of the set of all ordered pairs of disjoint coalitions, that is, the set 3^N and a worth function $b : 3^N \to \mathbb{R}$. For each $(S,T) \in 3^N$, the number $b(S,T)$ can be interpreted as the gain (whenever $b(S,T) > 0$) or loss (whenever $b(S,T) < 0$) that S can achieve when T is the opposer coalition and $N \setminus (S \cup T)$ is the neutral coalition. The pair $(\emptyset, N)$ represents the situation if all the players object to the change and $(N, \emptyset)$ represents the situation where all the players wish the change.

Definition 1 A bicooperative game is a pair (N, b), where N a finite set of players and $b : 3^N \to \mathbb{R}$ is a function such that $b(\emptyset, \emptyset) = 0$.

A bicooperative game $b \in \mathcal{BG}^N$ is *monotonic* if for all pairs $(S_1, T_1), (S_2, T_2)$ in 3^N with $(S_1, T_1) \sqsubseteq (S_2, T_2)$, we have $b(S_1, T_1) \leq b(S_2, T_2)$, that is, the addition of players to the *defender coalition* and the desertion of players from the *detractor coalition* does not decrease the worth. A *bicapacity* Grabisch and Labreuche (2005a,b) is a function $v : 3^N \to \mathbb{R}$ such that $v(\emptyset, \emptyset) = 0$ and $A \subseteq B \subseteq N$ implies $v(A, \cdot) \leq v(B, \cdot)$ and $v(\cdot, A) \geq v(\cdot, B)$. Although bicooperative games and bicapacities were proposed independently and for different domains, bicapacities are monotonic bicooperative games.

As for standard cooperative games, where each coalition $S \in 2^N$ can be identified with a $\{0,1\}$-vector, each $(S,T) \in 3^N$ can be identified with the $\{-1,0,1\}$-vector $\mathbf{1}_{(S,T)}$ defined, for all $i \in N$, by

$$\mathbf{1}_{(S,T)}(i) = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

In voting games, each voter has three choices: voting for a proposal, voting against it, and abstaining. Thus, only knowing who is in favor of the proposal is not enough to describe the situation. These games have been studied by Felsenthal and Machover (1997) under the name of *ternary voting games*. They generalize the standard voting games by recognizing abstention as an option alongside *yes* and *no* votes. They are formally described by mappings $u : 3^N \to \{-1, 1\}$ satisfying the following three conditions: $u(N, \emptyset) = 1$, $u(\emptyset, N) = -1$, and $\mathbf{1}_{(S,T)}(i) \leq \mathbf{1}_{(S',T')}(i)$ for all $i \in N$, implies $u(S, T) \leq u(S', T')$. A negative outcome, -1, is interpreted as defeat and a positive outcome, 1, as victory, the passing of a bill. More recently, Chua and Huang (2003) have studied the Shapley-Shubik index for ternary voting games.

More generally, one may imagine that each player can "participate" at different levels, ranging from no participation to full participation. If there is a finite number of such participation levels, we have a *multi-choice game* (see Hsiao and Raghavan 1993, Nouweland et al. 1995). When the degree of participation is defined on $[0, 1]$ we have a *fuzzy game* (see Aubin 1991, Tijs et al. 2004). In a multi-choice game, each player has at his/her disposal a linear ordered set of levels of participation labelled $0, 1, \ldots, m$ where 0 indicates no participation, and m full participation. A multi-choice game is a function $v : \{0, 1, \ldots, m\}^N \to \mathbb{R}$ such that $v(0, \ldots, 0) = 0$. The number $v(x)$ is the amount for profile of participation $x \in \{0, 1, \ldots, m\}^N$. Since $\{0, 1, 2\}^N$ is isomorphic to the set 3^N, the domains of bicooperative games and multi-choice games with $m = 2$ coincide. But the lattice structures of these sets are different. For instance, the element $(\emptyset, \emptyset)$ is central in the structure $(3^N, \sqsubseteq)$ and $(0, 0, 0)$ is the least element in $(3^N, \preceq)$, where $\preceq$ is the coordinatewise order. In a 2-choice game, the levels of participation are 0 (non participation), 1 (mild participation), and 2 (full participation). However, in a bicooperative game, the value 0 is central, and $-1, 1$ are symmetric extremes. This suggests that bicooperative games are a symmetrization of classical cooperative games, given by the lattice $\tilde{2}^N = \{(A, B) : A, B \in 2^N, \ A \cap B = \emptyset\}$ endowed with the order

$$(A, B) \sqsubseteq (C, D) \Longleftrightarrow A \subseteq C, \ B \supseteq D.$$

Throughout this paper, we use $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$ respectively. The number of players in S is denoted by $|S|$.

3 Solution concepts for bicooperative games

Since in a bicooperative game, $b(\emptyset, N)$ is the cost (or expense) incurred when all the players object to a proposal and $b(N, \emptyset)$ is the gain obtained when all players are in its favor, then the net profit is given by $b(N, \emptyset) - b(\emptyset, N)$. A solution concept for bicooperative games is a function that assigns to every bicooperative game a set of payoff vectors that distribute the net profit among the players. In this section, we introduce two solution concepts for bicooperative games: the *core* and the *Weber set*.

A vector $x \in \mathbb{R}^n$ which satisfies $\sum_{i \in N} x_i = b(N, \emptyset) - b(\emptyset, N)$ is an *efficient vector* and the set of all efficient vectors, denoted by $I^*(N, b)$, is the *preimputation set*, that is,

$$I^*(N,b) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = b(N, \emptyset) - b(\emptyset, N) \right\}.$$

The *imputations* for game b are the preimputations that satisfy the *individual rationality principle* for all players, that is, each player gets at least the difference between the amount that he can attain by himself against the rest of the players and the value of the pair $(\emptyset, N)$,

$$I(N,b) = \left\{ x \in I^*(N,b) : x_i \geq b(i, N \setminus i) - b(\emptyset, N) \quad \text{for all } i \in N \right\}.$$

The set $I(N, b)$ is determined by a finite number of inequalities and thus, it is a polyhedron. A polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$, where A is a matrix and b is a column vector, is bounded if and only if its characteristic cone $\{x \in \mathbb{R}^n : Ax \geq 0\} = \{0\}$ (see Schrijver 1986, Sect. 8.2). Clearly, $I(N, b)$ is bounded since the cone $\left\{x \in \mathbb{R}^n : \sum_{i \in N} x_i = 0, \ \ x_i \geq 0\right\} = \{0\}$.

We propose the following distribution criterion: every pair $(S, T) \in 3^N$ receives at least the amount it contributes to the pair $(\emptyset, N)$, the difference $b(S, T) - b(\emptyset, N)$. Now, two different sets of players contribute to the formation of each $(S, T) \in 3^N$. On the one hand, the players who are in $N \setminus T$ do not act against the players of S and so, they must receive a payoff (represented by a vector $z \in \mathbb{R}^n$). On the other hand, those players in $N \setminus T$ who also are in S must get an additional payoff (represented by a vector $y \in \mathbb{R}^n$). This leads us to the following definition of the core of a bicooperative game. Given $u \in \mathbb{R}^n$ and $S \subseteq N$ we denote by $u(S) = \sum_{i \in S} u_i$ with $u(\emptyset) = 0$.

Definition 2 Let $b \in \mathcal{BG}^N$. The core of b is the set

$$C(N,b) = \left\{ \begin{array}{l} x \in I^*(N,b) : \text{ there exist } y, z \in \mathbb{R}^n \text{ such that } x = y + z, \text{ and} \\ y(S) + z(N \setminus T) \geq b(S,T) - b(\emptyset, N), \text{ for all } (S,T) \in 3^N \end{array} \right\}.$$

Let $x \in C(N, b)$ and $i \in N$. Since $x_i = y_i + z_i \geq b(i, N \setminus i) - b(\emptyset, N)$, we obtain $x \in I(N, b)$, and hence $C(N, b)$ is a bounded set.

Let $x \in I^*(N, b)$ be such that $x = y + z$. Then

$$y(S) + z(N \setminus T) \geq b(S,T) - b(\emptyset, N) \Longleftrightarrow y(N \setminus S) + z(T) \leq b(N, \emptyset) - b(S,T).$$

Therefore, $C(N, b)$ is also the set of vectors $x \in I^*(N, b)$ such that there exist $y, z \in \mathbb{R}^n$ with $x = y + z$ and $y(N \setminus S) + z(T) \leq b(N, \emptyset) - b(S, T)$ for all $(S, T) \in 3^N$, that is, for each $(S, T) \in 3^N$ the payoff $y(N \setminus S)$ plus the payoff $z(T)$ must not exceed $b(N, \emptyset) - b(S, T)$, which is the amount that is foregone by forming the coalition (S,T) instead of the coalition $(N,\emptyset)$. Notice also that $x \in C(N,b)$ if and only if there exist $y, z \in \mathbb{R}^n$ such that $x = y + z$, and

$$y(S) + z(N \setminus T) \geq b(S,T) - b(\emptyset, N),$$
$$y(N \setminus S) + z(T) \leq b(N,\emptyset) - b(S,T),$$

for all $(S,T) \in 3^N$. These inequalities are similar in the bicooperative context to the inequalities characterizing the core in a cooperative game $v : 2^N \to \mathbb{R}$. Indeed

$$C(N,v)=\left\{x \in \mathbb{R}^n : x(S) \geq v(S)-v(\emptyset),\ \ x(N \setminus S) \leq v(N) - v(S), \quad \forall S \in 2^N\right\}.$$

Given a cooperative game (N,v) , each permutation $\pi = (i_1,\ldots,i_n)$ of the elements of N represents a sequential process of formation of the grand coalition N. The corresponding *marginal worth vector*, $a^{\pi}(v) \in \mathbb{R}^n$, gives the marginal contribution of every player to the coalition formed by his predecessors, that is, $a^{\pi}_{i_j}(v) = v\left(\pi\left(i_j\right)\right) - v\left(\pi\left(i_j\right) \setminus \left\{i_j\right\}\right)$ for all $i_j \in N$, where $\pi\left(i_j\right) = \left\{i_1,\ldots,i_j\right\}$ is the set of the predecessors of player i_j in the order π. The Weber set of game (N,v) is the convex hull of all marginal worth vectors.

In order to extend the idea of the Weber set to a bicooperative game (N,b) , it is assumed that all players think that $(N,\emptyset)$ is formed by a sequential process where at each step a player joins the defender coalition or a player leaves the detractor coalition. These sequential processes are obtained for each chain from $(\emptyset,N)$ to $(N,\emptyset)$. For each chain, a player can evaluate his contribution when he joins the defenders or when he leaves the detractors. These contributions are given as vectors in $\mathbb{R}^n$, called *superior marginal worth vectors* and *inferior marginal worth vectors*. To formalize this idea, we introduce the following notation.

For $N = \{1,\ldots,n\}$, let $\overline{N} = \{-n,\ldots,-1,1,\ldots,n\}$. Let $\Lambda : 3^N \to 2^{\overline{N}}$ be the isomorphism defined by $\Lambda(S,T) = S \cup \{-i : i \in N \setminus T\} \in 2^{\overline{N}}$, for each $(S,T) \in 3^N$. For instance, $\Lambda(\emptyset,N) = \emptyset$ and $\Lambda(N,\emptyset) = \overline{N}$. As $S \subseteq N \setminus T$, we see that $i \in \Lambda(S,T)$ and $i > 0$ imply $-i \in \Lambda(S,T)$.

In the lattice $\left(3^N, \sqsubseteq\right)$, let $\Theta\left(3^N\right)$ denote the set of all maximal chains going from $(\emptyset,N)$ to $(N,\emptyset)$. We identify a maximal chain $\theta \in \Theta\left(3^N\right)$ given by

$$(\emptyset,N) \sqsubset (S_1,T_1) \sqsubset \cdots \sqsubset \left(S_j,T_j\right) \sqsubset \cdots \sqsubset \left(S_{2n-1},T_{2n-1}\right) \sqsubset (N,\emptyset),$$

with an ordering $\theta = (i_1,\ldots,i_{2n})$ on $\overline{N}$ in such a way that $\Lambda\left(S_j,T_j\right) = \theta\left(i_j\right)$ for all $j = 1,\ldots,2n$, where $\theta\left(i_j\right) = \left\{i_1,\ldots,i_j\right\}$ is the set of predecessors of i_j in the order θ and its elements are written following the order of incorporation in the defender coalitions or desertion from the detractor coalitions. That is, if $i_j > 0$, i_j is the last player who joins S_j $\left(i_j \in S_j \text{ and } i_j \notin S_{j-1}\right)$ and, if $i_j < 0$, $-i_j$ is the last player who leaves T_{j-1} $\left(-i_j \notin T_j \text{ and } -i_j \in T_{j-1}\right)$. In particular $\Lambda^{-1}\left[\theta\left(i_{2n}\right)\right] = (N,\emptyset)$ and $\Lambda^{-1}\left[\theta\left(i_1\right) \setminus i_1\right] = (\emptyset,N)$.

For example, let $N = \{1,2,3\}$ and let $\theta \in \Theta\left(3^N\right)$ be given by

$$(\emptyset, N) \sqsubset (\emptyset, \{1,3\}) \sqsubset (\{2\}, \{1,3\}) \sqsubset (\{2\}, \{1\}) \sqsubset (\{2\}, \emptyset) \sqsubset (\{2,3\}, \emptyset) \sqsubset (N, \emptyset).$$

Its associated chain of sets in $2^{\overline{N}}$ is

$$\emptyset \subset \{-2\} \subset \{-2,2\} \subset \{-2,2,-3\} \subset \{-2,2,-3,-1\} \subset \{-2,2,-3,-1,3\} \subset \overline{N},$$

and the maximal chain can also be represented by the order $\theta = (-2,2,-3,-1,3,1)$.

Definition 3 Let $\theta \in \Theta\left(3^N\right)$ and $b \in \mathcal{BG}^N$. The inferior and superior marginal worth vectors with respect to θ are the vectors $m^\theta(b)$, $M^\theta(b) \in \mathbb{R}^n$ given by

$$m_i^\theta(b) = b\big(\Lambda^{-1}[\theta(-i)]\big) - b\big(\Lambda^{-1}[\theta(-i) \setminus -i]\big),$$
$$M_i^\theta(b) = b\big(\Lambda^{-1}[\theta(i)]\big) - b\big(\Lambda^{-1}[\theta(i) \setminus i]\big),$$

for all $i \in N$. The vector $a^\theta(b) = m^\theta(b) + M^\theta(b)$ is called the marginal worth vector with respect to θ.

We consider a bicooperative game (N, b) such that for all $S \subseteq N$, and for all $T, T' \subseteq N$ such that $S \cap T = S \cap T' = \emptyset$, we have $b(S,T) = b\left(S,T'\right)$. Then we define the cooperative game $v(S) = b(S,T)$ for all $T \subseteq N$ with $S \cap T = \emptyset$. First note that $v(\emptyset) = b(\emptyset, N) = b(\emptyset,\emptyset) = 0$ and $v(N) = b(N,\emptyset)$. We show that $C(N,b) = C(N,v)$. Indeed, if $x \in C(N,b)$ then there exist $y, z \in \mathbb{R}^n$ such that $x = y + z$ and $y(S) + z(N \setminus T) \geq b(S,T) - b(\emptyset, N)$, for all $(S,T) \in 3^N$. By setting $T = N \setminus S$, we obtain that for all $S \subseteq N$,

$$x(S) = y(S) + z(S) \geq b(S, N \setminus S) - b(\emptyset, N) = v(S).$$

Therefore $x \in C(N,v)$. Conversely, if $x \in C(N,v)$ then $x(S) \geq v(S)$ for all $S \subseteq N$ and $x(N) = v(N)$. We take $z = 0$ and $y = x$, and we obtain $x \in C(N,b)$. Furthermore, the marginal worth vectors of b coincide with those of the cooperative game v since $m_i^\theta(b) = 0$ and $M_i^\theta(b) = a^\pi(v)$, where π is the permutation of N given by θ restricted to N.

Proposition 4 *Let $b \in \mathcal{BG}^N$ and $\theta \in \Theta\left(3^N\right)$. Then,*

$$\sum_{j \in S} M_j^\theta(b) + \sum_{j \in N \setminus T} m_j^\theta(b) = b(S,T) - b(\emptyset, N),$$

for every (S,T) in the chain θ.

Proof Let $\theta \in \Theta\left(3^N\right)$ and (S,T) in θ be such that $\Lambda(S,T) = \{i_1, \ldots, i_{n+s-t}\}$, where $s = |S|, t = |T|, s + t \leq n$. Since $\theta\left(i_j\right) = \left\{i_1, \ldots, i_j\right\}$ for all $1 \leq j \leq n + s - t$, we obtain

$$\begin{aligned}\sum_{j\in S} M_j^{\theta}(b) + \sum_{j\in N\setminus T} m_j^{\theta}(b) &= \sum_{\{i_j\in\Lambda(S,T):i_j>0\}} M_{i_j}^{\theta}(b) + \sum_{\{i_j\in\Lambda(S,T):i_j<0\}} m_{-i_j}^{\theta}(b)\\ &= \sum_{i_j\in\Lambda(S,T)} \left[b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\right]\right) - b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\setminus i_j\right]\right)\right]\\ &= \sum_{j=1}^{n+s-t} \left[b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\right]\right) - b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\setminus i_j\right]\right)\right]\\ &= b(S,T) - b(\emptyset,N).\end{aligned}$$

□

Notice that $(S,T)=(N,\emptyset)$ implies $\sum_{j\in N}\left[M_j^{\theta}(b)+m_j^{\theta}(b)\right]=b(N,\emptyset)-b(\emptyset,N)$, and hence all the marginal worth vectors $a^{\theta}(b)$ are efficient.

Definition 5 Let $b\in\mathcal{BG}^N$. The Weber set of b is the convex hull of the marginal worth vectors of b, that is, $W(N,b)=\operatorname{conv}\left\{a^{\theta}(b):\theta\in\Theta\left(3^N\right)\right\}$.

Now we prove that the core of a bicooperative game is always included in its Weber set. The proof is closely related to the proof given by Derks (1992) of the parallel result for cooperative games.

Theorem 6 *If $b\in\mathcal{BG}^N$, then $C(N,b)\subseteq W(N,b)$.*

Proof Assume that there exists $x\in C(N,b)$ such that $x\notin W(N,b)$. Since $x\in C(N,b)$, then $\sum_{i\in N}x_i=b(N,\emptyset)-b(\emptyset,N)$, and there exist $y,z\in\mathbb{R}^n$ such that $x=y+z$ and $y(S)+z(N\setminus T)\geq b(S,T)-b(\emptyset,N)$ for all $(S,T)\in 3^N$.

Since $W(N,b)$ is convex and closed, by the Separation Theorem (see Rockafellar 1970), there exists $u\in\mathbb{R}^n$ such that

$$w\cdot u > x\cdot u \quad \text{for all } w\in W(N,b). \tag{1}$$

In particular, the above inequality holds for all marginal worth vectors $w=a^{\theta}(b)$ with $\theta\in\Theta\left(3^N\right)$. If the components of u are ordered in non-increasing order

$$u_{i_1}\geq u_{i_2}\geq\cdots\geq u_{i_{n-1}}\geq u_{i_n},$$

let $\theta\in\Theta\left(3^N\right)$ be the maximal chain given by $\theta=(-i_1,i_1,-i_2,i_2,\ldots,-i_n,i_n)$. Note that $\theta\left(i_j\right)\setminus i_j=\theta\left(-i_j\right)$ for all $1\leq j\leq n$, $\theta\left(-i_j\right)\setminus -i_j=\theta\left(i_{j-1}\right)$ for all

$2 \leq j \leq n$, and $\Lambda^{-1}[\theta(-i_1) \setminus -i_1] = (\emptyset, N)$. Then,

$$\begin{aligned}
a^{\theta}(b) \cdot u &= \sum_{j=1}^{n} a_{i_j}^{\theta}(b)\, u_{i_j} = \sum_{j=1}^{n} \left[M_{i_j}^{\theta}(b) + m_{i_j}^{\theta}(b)\right] u_{i_j} \\
&= \sum_{j=1}^{n} u_{i_j} \left[b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\right]\right) - b\left(\Lambda^{-1}\left[\theta\left(i_j\right) \setminus i_j\right]\right)\right] \\
&\quad + \sum_{j=1}^{n} u_{i_j} \left[b\left(\Lambda^{-1}\left[\theta\left(-i_j\right)\right]\right) - b\left(\Lambda^{-1}\left[\theta\left(-i_j\right) \setminus -i_j\right]\right)\right] \\
&= u_{i_n} b(N, \emptyset) + \sum_{j=1}^{n-1} u_{i_j} b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\right]\right) - u_{i_1} b(\emptyset, N) \\
&\quad - \sum_{j=2}^{n} u_{i_j} b\left(\Lambda^{-1}\left[\theta\left(i_{j-1}\right)\right]\right) \\
&= u_{i_n} b(N, \emptyset) - u_{i_1} b(\emptyset, N) + \sum_{j=1}^{n-1} \left(u_{i_j} - u_{i_{j+1}}\right) b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\right]\right) \\
&\leq u_{i_n} b(N, \emptyset) - u_{i_1} b(\emptyset, N) + \sum_{j=1}^{n-1} \left(u_{i_j} - u_{i_{j+1}}\right) \\
&\quad \times \left[\sum_{k=1}^{j} y_{i_k} + \sum_{k=1}^{j} z_{i_k} + b(\emptyset, N)\right] \\
&= u_{i_n} \left[\sum_{k=1}^{n} y_{i_k} + \sum_{k=1}^{n} z_{i_k} + b(\emptyset, N)\right] - u_{i_1} b(\emptyset, N) \\
&\quad + \sum_{j=1}^{n-1} \left(u_{i_j} - u_{i_{j+1}}\right) \left[\sum_{k=1}^{j} y_{i_k} + \sum_{k=1}^{j} z_{i_k} + b(\emptyset, N)\right] \\
&= \sum_{j=1}^{n} u_{i_j} \left(y_{i_j} + z_{i_j}\right) = \sum_{j=1}^{n} u_{i_j} x_{i_j} = u \cdot x,
\end{aligned}$$

which is in contradiction with (1). We conclude that $C(N, b) \subseteq W(N, b)$. □

An anonymous referee proposed the following model to analyze a bicooperative game (N, b). We consider an alternative set of players

$$\mathcal{N} = \{(i, t) : i \in N, \quad t \in \{1, 2\}\}.$$

A coalition $\mathcal{S} \subseteq \mathcal{N}$ is *feasible* if $(i, 2) \in \mathcal{S}$ implies $(i, 1) \in \mathcal{S}$. The set $\mathcal{F}$ of feasible coalitions, ordered by inclusion, is the distributive lattice $(\mathcal{F}, \cup, \cap)$ and there exists a lattice isomorphism $\Phi : (\mathcal{F}, \cup, \cap) \rightarrow (3^N, \vee, \wedge)$ given by $\Phi(\mathcal{S}) = (\mathcal{S}_2, N \setminus \mathcal{S}_1)$, where $\mathcal{S}_t = \{i \in N : (i,t) \in \mathcal{S}\},\ t \in \{1,2\}$. Note that $\mathcal{S}_2 \cap (N \setminus \mathcal{S}_1) = \emptyset \Longleftrightarrow \mathcal{S}_2 \subseteq \mathcal{S}_1$.

Since $\Phi(\emptyset) = (\emptyset, N)$, we can define the restricted cooperative game $v : \mathcal{F} \rightarrow \mathbb{R}$ as $v(\mathcal{S}) = b(\mathcal{S}_2, N \setminus \mathcal{S}_1) - b(\emptyset, N)$. Thus, there is coincidence between the core and the Weber set of (N, b) and the core and Weber set of the corresponding restricted game $(\mathcal{F}, v)$, and Theorem 6 is a direct consequence of Theorem 3.5 in Derks and Gilles (1995).

Although the model is formally similar, our approach focuses on the defender and detractor coalitions $(S, T) \in 3^N$ of the player set N. Therefore, we provide the definitions and direct results using our model. Bicooperative games are defined in combinatorial structures on the player set N and there exists a relationship with bisubmodular polyhedra. This fact will allow us in future research to apply some results of combinatorial optimization to the analysis of bicooperative games. Furthermore, classical solution concepts as the Shapley or Banzhaf values for cooperative games are defined for each player, and hence to extend them to bicooperative games it is more natural than to work with values for restricted games $(\mathcal{F}, v)$, where $\mathcal{F}$ is a set of feasible coalitions of the duplicated set of players $\mathcal{N}$.

4 Bisupermodular games

We now introduce a special class of bicooperative games.

Definition 7 A bicooperative game $b \in \mathcal{BG}^N$ is bisupermodular if, for all $(S_1, T_1), (S_2, T_2) \in 3^N$ we have

$$b((S_1, T_1) \vee (S_2, T_2)) + b((S_1, T_1) \wedge (S_2, T_2)) \geq b(S_1, T_1) + b(S_2, T_2),$$

or equivalently

$$b(S_1 \cup S_2, T_1 \cap T_2) + b(S_1 \cap S_2, T_1 \cup T_2) \geq b(S_1, T_1) + b(S_2, T_2).$$

The next proposition characterizes the bisupermodular games as those bicooperative games for which the marginal contribution of a player to a pair in 3^N is never less than the marginal contribution of this player to any pair contained in it. This characterization will be used in the proof of the following results.

Proposition 8 *The bicooperative game b is bisupermodular if and only if for all $i \in N$ and all (S_1, T_1), $(S_2, T_2) \in 3^{N \setminus i}$ such that $(S_1, T_1) \sqsubseteq (S_2, T_2)$, we have*

$$b(S_2 \cup i, T_2) - b(S_2, T_2) \geq b(S_1 \cup i, T_1) - b(S_1, T_1),$$
$$b(S_2, T_2) - b(S_2, T_2 \cup i) \geq b(S_1, T_1) - b(S_1, T_1 \cup i).$$

Proof

Necessary condition. Let (S_1,T_1), $(S_2,T_2) \in 3^{N\setminus i}$ be such that $(S_1,T_1) \sqsubseteq (S_2,T_2)$. Let $S_1' = S_1 \cup i$. Applying the definition of bisupermodularity to $\left(S_1',T_1\right)$ and (S_2,T_2), it follows that

$$b\left(S_1' \cup S_2, T_1 \cap T_2\right) + b\left(S_1' \cap S_2, T_1 \cup T_2\right) \geq b\left(S_1 \cup i, T_1\right) + b\left(S_2,T_2\right)$$

and hence

$$b\left(S_2 \cup i, T_2\right) + b\left(S_1,T_1\right) \geq b\left(S_1 \cup i, T_1\right) + b\left(S_2,T_2\right).$$

Analogously, taking $T_2' = T_2 \cup i$ and applying the definition of bisupermodularity to (S_1,T_1) and $\left(S_2,T_2'\right)$, it follows that

$$b\left(S_1, T_1 \cup i\right) + b\left(S_2,T_2\right) \geq b\left(S_1,T_1\right) + b\left(S_2, T_2 \cup i\right).$$

Sufficient condition. Let (S_1,T_1), $(S_2,T_2) \in 3^N$. If $(S_1,T_1) \sqsubseteq (S_2,T_2)$ or $(S_2,T_2) \sqsubseteq (S_1,T_1)$, the equality trivially holds. So, we consider the case $(S_1,T_1) \wedge (S_2,T_2) \neq (S_1,T_1)$ and $(S_1,T_1) \wedge (S_2,T_2) \neq (S_2,T_2)$. Let $\theta \in \Theta\left(3^N\right)$ be a maximal chain that contains (S_2,T_2) and $(S_1,T_1) \vee (S_2,T_2)$. Since $\Lambda\left(S_1,T_1\right) \setminus \Lambda\left(S_2,T_2\right) \neq \emptyset$, let $k = \left|\Lambda\left(S_1,T_1\right) \setminus \Lambda\left(S_2,T_2\right)\right|$ and so $\Lambda\left(S_1,T_1\right) \setminus \Lambda\left(S_2,T_2\right) = \{i_1,\ldots,i_k\}$, where the i_j are in the same order as they appear in the order θ, i.e.,

$$\Lambda^{-1}\left[\theta\left(i_1\right)\right] \sqsubset \Lambda^{-1}\left[\theta\left(i_2\right)\right] \sqsubset \cdots \sqsubset \Lambda^{-1}\left[\theta\left(i_k\right)\right].$$

Then, the chain θ is given by

$$\emptyset \subset \cdots \subset \Lambda\left(S_2,T_2\right) \subset \Lambda\left(S_2,T_2\right) \cup \{i_1\} \subset \cdots \subset \Lambda\left(S_2,T_2\right) \cup \{i_1,\ldots,i_k\} \subset \cdots \subset \overline{N}$$

or equivalently

$$(\emptyset, N) \sqsubset \cdots \sqsubset (S_2,T_2) \sqsubset \cdots \sqsubset (S_1,T_1) \vee (S_2,T_2) \sqsubset \cdots \sqsubset (N,\emptyset).$$

Let $A_0 = \emptyset$, $A_j = \left\{i_1,\ldots,i_j\right\}$, for all $1 \leq j \leq k$, and $(P,Q) = (S_1,T_1) \wedge (S_2,T_2)$. Then, we have

$$\Lambda^{-1}\left[\Lambda\left(P,Q\right) \cup A_j\right] \sqsubset \Lambda^{-1}\left[\Lambda\left(S_2,T_2\right) \cup A_j\right] \quad \text{for all } 1 \leq j \leq k.$$

We apply the hypothesis to $\Lambda^{-1}\left[\Lambda\left(P,Q\right) \cup A_j\right]$ and $\Lambda^{-1}\left[\Lambda\left(S_2,T_2\right) \cup A_j\right]$, and hence

$$\begin{aligned} &b\big(\Lambda^{-1}\left(\Lambda\left(P,Q\right) \cup A_j\right)\big) - b\big(\Lambda^{-1}\left(\Lambda\left(P,Q\right) \cup A_{j-1}\right)\big) \\ &\quad \leq b\big(\Lambda^{-1}\left(\Lambda\left(S_2,T_2\right) \cup A_j\right)\big) - b\big(\Lambda^{-1}\left(\Lambda\left(S_2,T_2\right) \cup A_{j-1}\right)\big) \end{aligned}$$

for all $1 \le j \le k$. Thus,

$$\begin{aligned}
&b\left((S_1,T_1)\right) - b\left((S_1,T_1) \wedge (S_2,T_2)\right) \\
&\quad = b\left(\Lambda^{-1}\left(\Lambda\left(P,Q\right) \cup A_k\right)\right) - b\left(P,Q\right) \\
&\quad = \sum_{j=1}^{k}\left[b\left(\Lambda^{-1}\left(\Lambda\left(P,Q\right) \cup A_j\right)\right) - b\left(\Lambda^{-1}\left(\Lambda\left(P,Q\right) \cup A_{j-1}\right)\right)\right] \\
&\quad \le \sum_{j=1}^{k}\left[b\left(\Lambda^{-1}\left(\Lambda\left(S_2,T_2\right) \cup A_j\right)\right) - b\left(\Lambda^{-1}\left(\Lambda\left(S_2,T_2\right) \cup A_{j-1}\right)\right)\right] \\
&\quad = b\left((S_1,T_1) \vee (S_2,T_2)\right) - b\left(S_2,T_2\right).
\end{aligned}$$

□

The following result allows the identification of the games for which the marginal worth vectors are in the core.

Theorem 9 *A bicooperative game $b \in \mathcal{BG}^N$ is bisupermodular if and only if all the marginal worth vectors of b are in the core $C(N,b)$.*

Proof

Necessary condition. Let $\theta \in \Theta\left(3^N\right)$. The marginal worth vectors are efficient and by Proposition 4, for every (S,T) in the chain θ, the marginal worth vector $a_i^\theta(b) = m_i^\theta(b) + M_i^\theta(b)$ satisfies

$$\sum_{j\in S} M_j^\theta(b) + \sum_{j\in N\setminus T} m_j^\theta(b) = b(S,T) - b(\emptyset,N).$$

We prove that for every pair (S,T) not in the chain θ,

$$\sum_{j\in S} M_j^\theta(b) + \sum_{j\in N\setminus T} m_j^\theta(b) \ge b(S,T) - b(\emptyset,N).$$

Indeed, let (S,T) be a pair in 3^N that does not belong to θ and such that $\Lambda(S,T) = \left\{i_1, i_2, \ldots, i_k\right\}$, $k = n + s - t$, where

$$\Lambda^{-1}\left[\theta\left(i_1\right)\right] \sqsubset \Lambda^{-1}\left[\theta\left(i_2\right)\right] \sqsubset \cdots \sqsubset \Lambda^{-1}\left[\theta\left(i_k\right)\right].$$

Let $A_0 = \emptyset$ and $A_j = \left\{i_1, i_2, \ldots, i_j\right\}$ for all $1 \le j \le k$. Note that, for all $1 \le j \le k$, we have $A_j = \Lambda(S,T) \cap \Lambda\left(\Lambda^{-1}\left[\theta\left(i_j\right)\right]\right)$, that is, $\Lambda^{-1}\left(A_j\right) = (S,T) \wedge \Lambda^{-1}\left[\theta\left(i_j\right)\right]$. Since b is bisupermodular, Proposition 8 implies that

$$b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\right]\right) - b\left(\Lambda^{-1}\left[\theta\left(i_j\right) \setminus i_j\right]\right) \ge b\left(\Lambda^{-1}\left(A_j\right)\right) - b\left(\Lambda^{-1}\left(A_{j-1}\right)\right),$$

for all $1 \leq j \leq k$, and we obtain

$$\begin{aligned}
\sum_{j\in S} M_j^{\theta}(b) + \sum_{j\in N\setminus T} m_j^{\theta}(b) &= \sum_{\{i_j\in\Lambda(S,T):i_j>0\}} M_{i_j}^{\theta}(b) + \sum_{\{i_j\in\Lambda(S,T):i_j<0\}} m_j^{\theta}(b) \\
&= \sum_{i_j\in\Lambda(S,T)} \left[b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\right]\right) - b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\setminus i_j\right]\right)\right] \\
&= \sum_{j=1}^{n+s-t} \left[b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\right]\right) - b\left(\Lambda^{-1}\left[\theta\left(i_j\right)\setminus i_j\right]\right)\right] \\
&\geq \sum_{j=1}^{n+s-t} \left[b\left(\Lambda^{-1}\left(A_j\right)\right) - b\left(\Lambda^{-1}\left(A_{j-1}\right)\right)\right] \\
&= b\left(S,T\right) - b\left(\emptyset,N\right).
\end{aligned}$$

Sufficient condition. For all $(S_1,T_1),(S_2,T_2) \in 3^N$, let $\theta \in \Theta\left(3^N\right)$ be a maximal chain which contains the pairs $(S_1,T_1)\wedge(S_2,T_2) = (S_1\cap S_2, T_1\cup T_2)$, and $(S_1,T_1)\vee(S_2,T_2) = (S_1\cup S_2, T_1\cap T_2)$. As the marginal worth vectors of (N,b) are elements of $C(N,b)$, we have that

$$\sum_{j\in S_k} M_j^{\theta}(b) + \sum_{j\in N\setminus T_k} m_j^{\theta}(b) \geq b\left(S_k,T_k\right) - b\left(\emptyset,N\right), \quad k=1,2.$$

By Proposition 4, the maximal chain θ also satisfies

$$\begin{aligned}
\sum_{j\in S_1\cap S_2} M_j^{\theta}(b) + \sum_{j\in N\setminus(T_1\cup T_2)} m_j^{\theta}(b) &= b\left((S_1,T_1)\wedge(S_2,T_2)\right) - b\left(\emptyset,N\right), \\
\sum_{j\in S_1\cup S_2} M_j^{\theta}(b) + \sum_{j\in N\setminus(T_1\cap T_2)} m_j^{\theta}(b) &= b\left((S_1,T_1)\vee(S_2,T_2)\right) - b\left(\emptyset,N\right).
\end{aligned}$$

Therefore, $b(S_1,T_1) + b(S_2,T_2) - 2b(\emptyset,N)$

$$\begin{aligned}
&\leq \sum_{j\in S_1} M_j^{\theta}(b) + \sum_{j\in N\setminus T_1} m_j^{\theta}(b) + \sum_{j\in S_2} M_j^{\theta}(b) + \sum_{j\in N\setminus T_2} m_j^{\theta}(b) \\
&= \sum_{j\in S_1\cup S_2} M_j^{\theta}(b) + \sum_{j\in S_1\cap S_2} M_j^{\theta}(b) + \sum_{j\in N\setminus(T_1\cup T_2)} m_j^{\theta}(b) + \sum_{j\in N\setminus(T_1\cap T_2)} m_j^{\theta}(b) \\
&= b\left((S_1,T_1)\wedge(S_2,T_2)\right) + b\left((S_1,T_1)\vee(S_2,T_2)\right) - 2b\left(\emptyset,N\right).
\end{aligned}$$

Hence, $b(S_1,T_1) + b(S_2,T_2) \leq b\left((S_1,T_1)\wedge(S_2,T_2)\right) + b\left((S_1,T_1)\vee(S_2,T_2)\right).$

□

Since the core of a bicooperative game $b \in \mathcal{BG}^N$ is a convex set, a consequence of this theorem is the following characterization.

Corollary 10 *A bicooperative game* $b \in \mathcal{BG}^N$ *is bisupermodular if and only if* $W(N,b) = C(N,b)$.

Note that the bicooperative game (N,b) is bisupermodular if and only if the restricted game $(\mathcal{F},v)$, defined in the third section is convex. Taking into account the relation between the lattices $\left(3^N, \vee, \wedge\right)$ and $(\mathcal{F}, \cup, \cap)$, and hence, the relation between the concepts of the core and the Weber set of the bicooperative game (N,b) and the corresponding sets of the restricted game $(\mathcal{F},v)$, the above results are a direct consequence of Theorem 4.2 in Derks and Gilles (1995).

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