

COMPLEXITY IN COOPERATIVE GAME THEORY

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Abstract. We introduce cooperative games (N, v) with a description polynomial in n , where n is the number of players. In order to study the complexity of cooperative game problems, we assume that a cooperative game $v : 2^N \rightarrow \mathbb{Q}$ is given by an oracle returning $v(S)$ for each query $S \subseteq N$. Finally, we consider several cooperative game problems and we give a list of complexity results.

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1 Introduction

Let $v : 2^N \rightarrow \mathbb{Q}$ be a cooperative game with rational worths. The coalition description of v is given by its 2^n coalitional values $\{v(S) : S \subseteq N\}$. If the complexity is measured in the size of these values, then the input is exponential in n , and thus, the most of the complexity questions would become easy. Therefore, let us start with games (N, v) with a description polynomial in n , where n is the number of players.

Example: A *weighted voting game* $[q; w_1, \dots, w_n]$, where w_i is the number of votes of each player i , is a description polynomial in n of the game v , defined by

$$v(S) = \begin{cases} 1, & \text{if } w(S) \geq q \\ 0, & \text{if } w(S) < q, \end{cases}$$

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where $w(S) = \sum_{i \in S} w_i$. We will denote these games by wVG .

Example: Deng and Papadimitriou [1] studied computational complexity of cooperative solution concepts for the following game. If $G = (N, E)$ is a graph with weight $w : E \rightarrow \mathbb{Z}$, the *weighted graph game* $v_G : 2^N \rightarrow \mathbb{Z}$ is given by $v_G(S) = \sum_{e \in G[S]} w(e)$, and the class of these games is denoted by wGG .

The game v_G evaluates the profit $v_G(S)$ of any coalition S as the profit of the subgraph of G induced by the vertex set S . We can use it for the fair division between n cities of the income from a communication network connecting them. Since $|E| \leq \frac{1}{2}n(n-1)$, this game has a description polynomial in n .

Example: Nagamochi, Zeng, Kabutoya and Ibaraki [14] consider a matroid (E, \mathcal{M}) with weight function $w : E \rightarrow \mathbb{Q}$. The *minimum base game* $c : 2^E \rightarrow \mathbb{Q}$ is given by

$$c(S) = \min \left\{ \sum_{e \in B(S)} w(e) : B(S) \text{ is a basis of } S \right\},$$

and we denoted by MBG the class of these games. If the matroid (E, \mathcal{M}) is the forest matroid of a graph $G = (N, E)$, then the *minimum forest game* is

$$c(S) = \min \left\{ \sum_{e \in F_S} w(e) : F_S \text{ is a maximal forest on } S \right\},$$

for all $S \subseteq E$ and the class is MFG .

For the last game, we define the encoding length or size of a matroid (E, \mathcal{M}) as the number of its independents sets $|\mathcal{M}|$ and we assume that the matroid is given by an *oracle* that testing if a given set $S \subseteq E$ is independent or not. In this context, a matroid on $|E|$ elements can be represented by a string of length $2^{|E|}$, where each character indicates whether a subset $S \subseteq E$ is independent.

In this model, a minimum base $B(S)$ on S can be obtained, in polynomial time, by the greedy algorithm as follows:

Order $S = \{e_1, \dots, e_s\}$ so that $c(e_1) \leq \dots \leq c(e_t) < 0 \leq \dots \leq c(e_s)$;
 $B(S) \leftarrow \emptyset$;

For $i = 1$ to s do

If $B(S) \cup \{e_i\} \in \mathcal{M}$ then $B(S) \leftarrow B(S) \cup \{e_i\}$.

2 Cooperative game problems

In order to study the complexity of *cooperative game problems*, we assume that a cooperative game $v : 2^N \rightarrow \mathbb{Q}$ is given by an oracle returning $v(S)$ for each query $S \subseteq N$. For this oracle we have a polynomial p_v such that for every input of size at most n , the answer of the oracle has size at most $p_v(n)$. Therefore, we know an upper bound $\beta \geq \max \{\text{size}(f(S)) : S \subseteq N\}$.

Definition 1 Let $v : 2^N \rightarrow \mathbb{Q}$ be a game given by an oracle with upper bound β . The size of v is defined by $\text{size}(v) = |N| + \beta$.

Definition 2 We say that a problem for a game $v : 2^N \rightarrow \mathbb{Q}$ given by an oracle, is solvable in oracle-polynomial time if the number of computational steps, counting each call to the oracle as one step, is polynomially bounded in $\text{size}(v)$.

Example: A *min-cost spanning tree game* is defined by a set $N = \{1, \dots, n\}$ of players, a supply vertex 0, a complete graph $G = K_{n+1} = (N \cup \{0\}, E)$ and a nonnegative edge-weight function $w : E \rightarrow \mathbb{Q}_+$. The cost $c(S)$ of any coalition $S \subseteq N$, is defined by

$$c(S) = \min \left\{ \sum_{e \in T_S} w(e) : T_S \text{ is a spanning tree of } G[S \cup \{0\}] \right\}.$$

The class of these games is denoted by *MSTG*.

Remark 1 We know that the greedy algorithm is a polynomial-time algorithm for the min-cost spanning tree problem. Therefore, the oracle is any algorithm that computes $c(S)$ in polynomial time with respect to $|N| + \text{size}(w)$, for all $S \subseteq N$.

We consider now several cooperative game problems and we give a list of complexity results.

Problem: MEMBER CORE

- Instance: A game $v : 2^N \rightarrow \mathbb{Q}$ with $\text{size}(v)$, and $x \in I(v) \cap \mathbb{Q}^N$.
- Question: Is x an element of the core of v ?

Good News: For the class wGG , Deng and Papadimitriou [1] proved that it is in P if the weight function w is nonnegative. If v is convex (or concave) then it can be solved in oracle-polynomial time by the greedy algorithm (Grötschel, Lovász and Schrijver [9, Section 10.2]).

Bad News: coNP-complete for the classes wGG and $MSTG$ (Deng and Papadimitriou [1] and Faigle, Kern, Fekete and Hochstättler [2]).

Comments: Deng and Papadimitriou used reduction from MAX-CUT and Faigle *et al.* [2] used reduction from EXACT COVER BY 3-SETS (Papadimitriou [15, page 201]). It is still open the complexity for the classes MBG and MFG .

Problem: CORE

- Instance: A game $v : 2^N \rightarrow \mathbb{Q}$ with size (v) .
- Question: Is the core of v empty?

Good News: For wGG with nonnegative weights and MFG with no all-negative circuit is in P. For MBG such that the underlying matroid has no all-negative circuits it is solvable in oracle-polynomial time (Deng and Papadimitriou [1] and Nagamochi *et al.* [14]).

Bad News: NP-complete for wGG (Deng and Papadimitriou [1]).

Problem: SUBADDITIVITY

- Instance: A game $c : 2^N \rightarrow \mathbb{Q}$ with size (v) .
- Question: Is the game c subadditive?

Good News: Nagamochi *et al.* [14] obtained an $O(\text{size}(v))$ time algorithm for the class MFG and proved that it is solvable in oracle linear time for MBG .

Problem: SUBMODULARITY

- Instance: A game $c : 2^N \rightarrow \mathbb{Q}$ with size (v) .
- Question: Is the game c submodular?

Good News: It is solvable in oracle-polynomial time for the class *MBG* and in $O(\text{size}(v))$ time for *MFG* (Nagamochi *et al.* [14]).

Problem: SHAPLEY VALUE

- **Instance:** A game $v : 2^N \rightarrow \mathbb{Q}$ with $\text{size}(v)$.
- **Solution:** The Shapley value $\Phi(v)$.

Good News: Megiddo [13] showed that it can be found in $O(n)$ time for *MSTG*, in the special case where the underlying graph is a tree. For *wGG*, Deng and Papadimitriou [1] proved that it can be computed in $O(n^2)$.

Bad News: For the class *wVG*, Deng and Papadimitriou [1] proved that is $\#P$ -complete, that is, as difficult as any counting problem in NP. Nagamochi *et al.* [14] proved that is $\#P$ -complete for the class *MFG* and that there is no oracle-polynomial time algorithm for the class *MBG*.

Comments: Deng and Papadimitriou [1] used the counting problem associated with *KNAPSACK* (Papadimitriou [15, page 202]). Nagamochi *et al.* [14] used the $\#P$ -completeness of *GRAPH RELIABILITY* (Valiant [18]).

Problem: TIJS VALUE

- **Instance:** A game $v : 2^N \rightarrow \mathbb{Q}$ with $\text{size}(v)$.
- **Solution:** The Tijs value $\tau(v)$.

Good News: For the class *MFG*, Nagamochi *et al.* [14] proved that it can be computed in $O(m^2 + mn \log_2 n)$ time, where $n = |N|$ and $m = |E|$.

Bad News: For the class *MBG*, there is no oracle-polynomial time algorithm.

Comments: Nagamochi *et al.* [14] used the Dijkstra's algorithm implemented by Fredman and Tarjan [5] (see also Lovász *et al.* [12, Theorem 3.3]).

Problem: NUCLEOLUS

- **Instance:** A game $v : 2^N \rightarrow \mathbb{Q}$ with $\text{size}(v)$.
- **Solution:** The nucleolus $\eta(v)$.

Good News: Granot and Granot [6] and Granot, Maschler, Owen and Zhu [7] showed that it can be found in strongly polynomial time for a subclass of *MSTG*. Solymosi and Raghavan [16] provides an $O(n^4)$ algorithm for the class of assignments games. For the class *wGG*, Deng and Papadimitriou [1] proved that it can be computed in $O(n^2)$. If v is convex (or concave) then it can be solved in oracle-polynomial time (Kuipers [10] and Faigle, Kern and Kuipers [3]). The \mathcal{B} -nucleolus can be calculated in strongly polynomial time for games with size (\mathcal{B}) polynomially bounded in n (Granot, Granot and Zhu [8], Kuipers, Solymosi and Aarts [11], Solymosi, Aarts and Driessen [17]).

Bad News: NP-hard, that is, at least as difficult as any optimization problem in NP, for the class *MSTG* (Faigle, Kern and Kuipers [4]).

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