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Values and potential of games with cooperation structure

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Abstract. Games with cooperation structure are cooperative games with a family of *feasible coalitions*, that describes which coalitions can negotiate in the game. We study a model of *cooperation structure* and the corresponding restricted game, in which the feasible coalitions are those belonging to a *partition system*. First, we study a recursive procedure for computing the Hart and Mas-Colell potential of these games and we develop the relation between the dividends of Harsanyi in the restricted game and the worths in the original game. The properties of *partition convex geometries* are used to obtain formulas for the *Shapley* and *Banzhaf values* of the players in the restricted game $v^{\mathscr{L}}$, in terms of the original game v. Finally, we consider the Owen multilinear extension for the restricted game.

Key words: Shapley value, Hart and Mas-Colell potential, convex geometry

1. Cooperation structure

A *cooperative game* is a pair (N, v), where N is a finite set and $v : 2^N \to \mathbb{R}$, is a function with $v(\emptyset) = 0$. The elements of $N = \{1, 2, \ldots, n\}$ are called *players*, the subsets $S \in 2^N$ *coalitions* and $v(S)$ is the *worth* of S. By Γ^N we denote the set of all games (N, v). We will use a shorthand notation and write $S \cup i$ for the set $S \cup \{i\}$, and $S \backslash i$ for $S \backslash \{i\}$. The *Shapley value* for the player $i \in N$ is defined by

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$$\Phi_i(N,v) = \sum_{\{S \subseteq N \mid i \in S\}} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S \setminus i)], \tag{1}$$

where $n = |N|$, $s = |S|$. This value is an average of *marginal contributions* $v(S) - v(S \setminus i)$ of a player i to all possible coalitions $S \in 2^N \setminus \{\emptyset\}$. In this value, the sets S of different size get different weight. Dubey and Shapley [3] suggested the following *Banzhaf value*,

$$\beta_i'(N,v) = \sum_{\{S \subseteq N \mid i \in S\}} \frac{1}{2^{n-1}}[v(S) - v(S \setminus i)], \quad i \in N. \tag{2}$$

> "This definition enjoys the symmetry, dummy, and linearity properties that are traditionally used to axiomatize the Shapley value. Only the *efficiency* axiom fails, since we have in general $\sum_{i \in N} \beta_i'(N,v) \neq v(N)$".

In cooperative game theory it is generally assumed that the whole group of players decides to cooperate. However, the classical model seems to be inappropriate in modelling certain situations. So, the hypothesis of the Shapley value (the probability of a coalition depend on its size, with the total probability of each size being the same) or the Banzhaf value (all coalitions are equally possible) maybe unrealistic. Because of this, Myerson [11, p. 444] proposed:

> "the term *cooperation structure* to refer to any mathematical structure that describes which coalitions (within the set of all $2^n - 1$ possible coalitions) can negotiate or coordinate effectively in coalition game".

Myerson [9] introduced the *graph-restricted* games and the *Myerson value*. These games and this value were also investigated in Owen [14], who obtained a formula for computing the dividends in restricted games, where the graph is a tree. Borm, Owen and Tijs [1] defined the *position value* for communication situations and provided a new axiomatic characterization of the Myerson value.

In this paper, a general cooperation structure is considered, which is an extension of the graph-restricted games. A *set system* on a finite ground set N is a pair $(N, \mathscr{F})$, with $\mathscr{F} \subseteq 2^N$. The sets belonging to $\mathscr{F}$ are called *feasible coalitions*. For any $S \subseteq N$, maximal feasible subsets of S are called *components* of S.

Definition 1. *A partition system is a pair $(N, \mathscr{F})$ satisfying the following properties:*

(**P1**) $\emptyset \in \mathscr{F}$, *and* $\{i\} \in \mathscr{F}$ *for every* $i \in N$.
(**P2**) *For all $S \subseteq N$, the components of S, denoted by $\Pi_S = \{T_1, \ldots, T_p\}$, form a partition of S.*

Proposition 1. *A set system $(N, \mathscr{F})$ which satisfies property $(P1)$ is a partition system if and only if $F \in \mathscr{F}$, $G \in \mathscr{F}$ and $F \cap G \neq \emptyset$ imply $F \cup G \in \mathscr{F}$.*

Proof: Suppose $F \cup G \notin \mathscr{F}$ for some pair $\{F, G\} \subset \mathscr{F}$ with $F \cap G \neq \emptyset$. Then, there exist two components $\{T_1, T_2\}$ of $F \cup G$, such that $T_1 \supseteq F$ and $T_2 \supseteq G$. Hence, $T_1 \cap T_2 \supseteq F \cap G \neq \emptyset$, which contradicts property (P2).

Conversely, if $(N, \mathscr{F})$ satisfies property (P1), then $S = \bigcup_{i \in S}\{i\}$ for every $S \subseteq N$. Let $\Pi_S = \{T_1, \ldots, T_p\}$ be the family of components of S. If Π_S is not a partition of S, then $T_i \cap T_j \neq \emptyset$, and hence $T_i \cup T_j \in \mathscr{F}$, which contradicts the maximality of T_i and T_j. □

Example 1: The following collections of subsets of N, given by $\mathscr{F} = \{\emptyset, \{1\}, \ldots, \{n\}\}$, and $\mathscr{F} = 2^N$, are the minimal and maximal partition systems.

Example 2: In a sequencing situation there is a queue, consisting of n customers waiting to be served at a counter. Curiel, Pederzoli and Tijs [2] introduced *sequencing games* (N, v), defined by $v(S) := \sum\{v(T) | T \in \Pi_S\}$, where $v(T)$ is equal to the maximal cost savings the coalition can obtain by rearranging their positions in the queue. The components of Π_S are the maximal intervals of S in a total order on N. If (N, v) is a sequencing game then $(N, \leq)$ is a chain and the collection $\mathscr{F} = \{T \subseteq N | T \text{ is an interval of} N\}$, is a partition system.

Example 3: A communication situation is a triple (N, G, v), where (N, v) is a game and $G = (N, E)$ is a graph. This concept was first introduced by Myerson [9], and investigated by Owen [14] and Borm, Owen and Tijs [1]. In this model, the set system

$$\mathscr{F} = \{S \subseteq N | (S, E(S)) \text{ is a connected subgraph of } G\},$$

is a partition system.

Example 4: A hypergraph communication situation is a triple $(N, \mathscr{H}, v)$, where (N, v) is a game and $\mathscr{H} \subseteq 2^N$ is a hypergraph. The idea of modelling communication by means of *conferences* $H \in \mathscr{H}$ is due to Myerson [10]. The collection of the *interaction sets* (see van den Nouweland, Borm and Tijs [13]) plus the empty set, is a concept equivalent to that of partition system.

2. Restricted games

Let $(N, \mathscr{F})$ be a partition system. The $\mathscr{F}$-restricted game associated to (N, v) is the game $(N, v^{\mathscr{F}})$, defined by

$$v^{\mathscr{F}}(S) = \sum\{v(T) | T \in \Pi_S\},$$

where Π_S is the collection of the components of $S \subseteq N$. If $S \in \mathscr{F}$ then $v^{\mathscr{F}}(S) = v(S)$.

Note that if the partition system $(N, \mathscr{F})$ is defined by a communication situation, the restricted game is called a *graph-restricted* game. The map $L_{\mathscr{F}} : \Gamma^N \to \Gamma^N$, defined by $L_{\mathscr{F}}(v) = v^{\mathscr{F}}$, is a linear operator.

Remark 1: If $\mathscr{F}$ is the partition system of Example 3, then the game $v^{\mathscr{F}}$ is a Γ-component additive game which are studied by Potters and Reijnierse [16].

Unanimity games are considered. For any $T \subseteq N, T \neq \emptyset$,

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise,} \end{cases}$$

is called the T-unanimity game. Every game is a linear combination of unanimity games,

$$v = \sum_{T \subseteq N} \Delta_T(v) u_T, \quad \text{where } \Delta_T(v) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S).$$

Following Harsanyi [6], we shall call $\Delta_T(v)$ the *dividend* of T in the game v. The linearity implies that

$$v^{\mathscr{F}} = \sum_{T \subseteq N} \Delta_T(v) u_T^{\mathscr{F}}, \tag{3}$$

where the game $u_T^{\mathscr{F}}$ satisfies

$$u_T^{\mathscr{F}}(S) = \sum_{F \in \Pi_S} u_T(F) = \begin{cases} 1, & \text{if there exists } F \in \mathscr{F} \text{ such that } T \subseteq F \subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

Owen [14, Theorems 2 and 3], gave the following result: *The unanimity games u_T, where T is connected in the graph G, form a basis of the graph-restricted games.* We shall obtain a similar property for partition systems.

Theorem 1. *If $(N, \mathscr{F})$ is a partition system then the unanimity games $\{u_T | T \in \mathscr{F}, T \neq \emptyset\}$ form a basis of the vector space $\{(N, v^{\mathscr{F}}) | v \in \Gamma^N\}$, i.e.,*

$$v^{\mathscr{F}} = \sum_{T \in \mathscr{F}} \Delta_T(v^{\mathscr{F}}) u_T, \text{ where } \Delta_{\emptyset}(v^{\mathscr{F}}) = 0.$$

Proof: Every game $(N, v^{\mathscr{F}})$ is uniquely determined by the values $\{v(S) | S \in \mathscr{F}, S \neq \emptyset\}$. Then, the vector space of these games will be identified with $\mathbb{R}^{|\mathscr{F}|-1}$. The unanimity game $u_T^{\mathscr{F}} = u_T$ if and only if $T \in \mathscr{F}$, hence the subset $\{u_T | T \in \mathscr{F}, T \neq \emptyset\}$ contains $|\mathscr{F}| - 1$ games of the type $v^{\mathscr{F}}$ and it is lineary independent. Therefore, it is a basis. □

3. Hart and Mas-Colell potential for restricted games

The potential function for cooperative games was defined by Hart and Mas-Colell [7]. Given a game (N, v) and a coalition $S \subseteq N$, the *subgame* (S, v) is obtained by restricting v to 2^S. Let Γ denote the set of all games. The potential is a function $P : \Gamma \to \mathbb{R}$ which assigns to each game (N, v) a real number $P(N, v)$ and satisfies the following equations

$$P(\emptyset, v) = 0, \quad P(S, v) = \frac{1}{|S|}\left[v(S) + \sum_{i \in S} P(S \backslash \{i\}, v)\right], S \in 2^N \backslash \{\emptyset\}. \tag{4}$$

Moreover, the *marginal contribution* of a player i coincides with the Shapley value:

$$P(N, v) - P(N \backslash \{i\}, v) = \Phi_i(N, v), \quad \forall i \in N.$$

There are two explicit formulas for the potential (see [7, Prop. 1 and 2]),

$$P(N, v) = \sum_{S \subseteq N} \frac{\Delta_S(v)}{|S|},$$

$$P(N, v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S), \quad n = |N|,\ s = |S|.$$

Definition 2. *Let* $(N, \mathscr{F})$ *be a partition system. The* $\mathscr{F}$*-restricted potential of the game* (N, v) *is defined by* $P(N, v^{\mathscr{F}})$.

The recursive procedure defined by the formula (4) implies an algorithm for computing $P(N, v^{\mathscr{F}})$. A new algorithm, in terms of v, is stated in the next theorem.

Theorem 2. *The restricted potential* $P(N, v^{\mathscr{F}})$ *satisfies:*

$$P(S, v^{\mathscr{F}}) = \frac{1}{|S|}\left[v(S) + \sum_{i \in S} P(S \backslash \{i\}, v^{\mathscr{F}})\right], \quad \textit{for all } S \in \mathscr{F}.$$

$$P(S, v^{\mathscr{F}}) = \sum \{P(S_k, v^{\mathscr{F}}) |\ S_k \in \Pi_S\}, \quad \textit{for all } S \notin \mathscr{F}.$$

Proof: If $S \in \mathscr{F}$, then $v^{\mathscr{F}}(S) = v(S)$. Let $S \notin \mathscr{F}$. It follows from [7, Prop. 1] that

$$P(S, v^{\mathscr{F}}) = \sum_{T \subseteq S} \frac{\Delta_T(v^{\mathscr{F}})}{|T|}.$$

By Theorem 1 we know that $\Delta_T(v^{\mathscr{F}}) = 0$ unless $T \in \mathscr{F}$, hence

$$P(S, v^{\mathscr{F}}) = \sum_{\{T \in \mathscr{F} | T \subseteq S\}} \frac{\Delta_T(v^{\mathscr{F}})}{|T|}.$$

Since $S \notin \mathscr{F}$, property (P2) implies that $S = \bigcup_{k=1}^{p} S_k$, where $\Pi_S = \{S_1, \ldots, S_p\}$ is the collection of components of S. Then, we have the partition

$$\{T \in \mathscr{F} \mid T \subseteq S\} = \bigcup_{k=1}^{p} \{T \in \mathscr{F} | T \subseteq S_k\}.$$

This implies that

$$P(S, v^{\mathscr{F}}) = \sum_{k=1}^{p} \left[\sum_{\{T \in \mathscr{F} \mid T \subseteq S_k\}} \frac{\Delta_T(v^{\mathscr{F}})}{|T|} \right] = \sum_{k=1}^{p} P(S_k, v^{\mathscr{F}}). \qquad \square$$

4. Convex geometries

Convex geometries are a combinatorial abstraction of convex sets introduced by Edelman and Jamison [4].

Definition 3. *The finite set system $(N, \mathscr{L})$ is a convex geometry on N if it satisfies the properties:*

(C1) $\emptyset \in \mathscr{L}$,
(C2) *$\mathscr{L}$ is closed under intersections,*
(C3) *If $C \in \mathscr{L}$ and $C \neq N$, then there exists $j \in N \backslash C$ such that $C \cup j \in \mathscr{L}$.*

Property (C2) implies that intersections of feasible coalitions should also be feasible, since the players agree on a profile of cooperation. In the model of *conference structures* by Myerson [10], two players are connected if they can be coordinated by meeting in separate conferences which have some members in common to serve as intermediaries. In our model, the coalitions of intermediaries are in the cooperation structure.

A maximal chain of $\mathscr{L} \subseteq 2^N$ is an ordered collection of convex sets that is not contained in any larger chain. From property (C3) and by induction, Edelman and Jamison [4] showed that every maximal chain contains $n+1$ convex sets

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{n-1} \subset S_n = N,$$

and the cardinal $|S_k| = k$, for all $k = 0, 1, \ldots, n$. Thus, the *hierarchical situations* by Moulin [8], when users pay their incremental costs according to an ordering of N, can be modeled by convex geometries.

For any subset S of N we define the *closure* of S, denoted by $\overline{S}$, to be

$$\overline{S} := \bigcap \{C | C \in \mathscr{L}, C \supseteq S\}.$$

The map $\overline{} : 2^N \to 2^N$ is a *closure operator* [18, p. 159], with the additional condition that $\overline{\emptyset} = \emptyset$. The subsets in the collection $\mathscr{L}$ or, equivalently, those subsets of N such that $\overline{S} = S$, will be called *convex sets*. Every convex geometry $(N, \mathscr{L})$ satisfies the anti-exchange property (see Edelman and Jamison [4]),

$$\forall S \subseteq N,\ i, j \notin \overline{S},\ j \in \overline{S \cup i} \Rightarrow i \notin \overline{S \cup j}.$$

This property is a combinatorial abstraction of the convex closure in Euclidean spaces. That is, in Figure 1, the points x and y are not in the convex hull of the set S. If y is in $\operatorname{conv}(S \cup x)$ then x is outside $\operatorname{conv}(S \cup y)$.

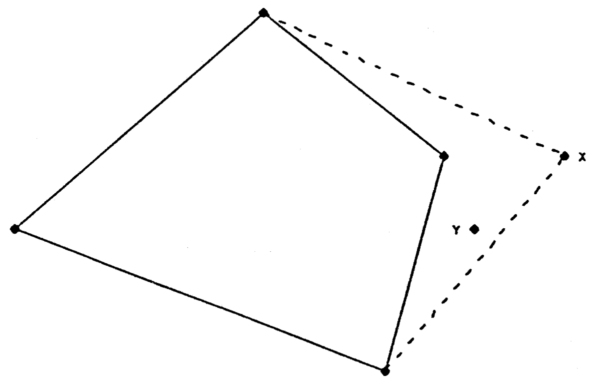

Fig. 1. The anti-exchange property

An element i of a convex set $C \in \mathscr{L}$ is an *extreme point* of C if $C \setminus i \in \mathscr{L}$. The set of extreme points of C is denoted by $ex(C)$. The convex geometries are the abstract closure spaces satisfying the finite Minkowski-Krein-Milman property: *Every convex set is the closure of its extreme points.*

Definition 4. *A partition convex geometry is a convex geometry* $(N, \mathscr{L})$ *which satisfies properties (P1) and (P2).*

In the following it will be neccesary several concepts of graph theory. A graph $G = (N, E)$ is connected if any two vertices can be joined by a path. A maximal connected subgraph of G is a *component* of G. A *cutvertex* is a vertex whose removal increases the number of components, and a *bridge* is an edge with the same property. A graph is *2-connected* if it is connected, has at least 3 vertices and contains no cutvertex. A subgraph B of a graph G is a *block* of G if either B is a bridge or else it is a maximal 2-connected subgraph of G.

A graph G is a *block graph* if every block is a complete graph. The block graphs are called *cycle-complete* graphs in van den Nouweland and Borm [12]. If G is a disjoint union of trees, then G is a block graph. Jamison [4, Th. 3.7] showed: $G = (N, E)$ *is a connected block graph if and only if the collection of subsets of* N *which induce connected subgraphs is a convex geometry.*

Example 5: Let $G = (N, E)$ be a connected block graph. In this situation, the family

$$\mathscr{L} = \{S \subseteq N | (S, E(S)) \text{ is a connected subgraph of } G\},$$

is a partition convex geometry.

Example 6: A subset S of a poset $(P, \leq)$ is *convex* whenever $a \in S, b \in S$ and $a \leq b$ imply $[a, b] \subseteq S$. The convex subsets of any poset P form a closure system which is denoted by $C_o(P)$. If $C \in C_o(P)$ then $ex(C)$ is the union of the maximal and minimal elements of C. Moreover, $C_o(P)$ is a convex geometry. Edelman [5] studied *voting games* such that the feasible coalitions are the convex sets of $C_o(P)$, where P is the chain defined by the policy order (see Figure 2).

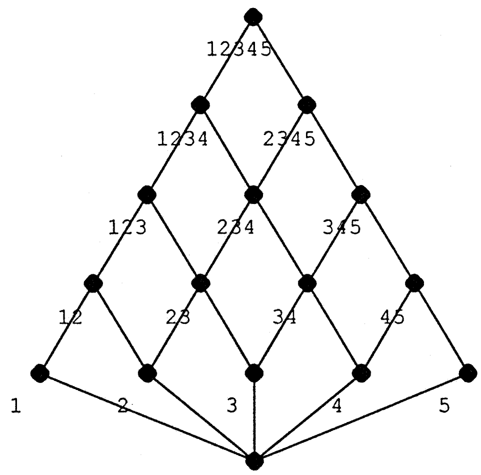

Fig. 2. The convex geometry $\mathrm{Co}(\{1 < 2 < 3 < 4 < 5\})$

Let $(N, \mathscr{L})$ be the partition convex geometry of subsets of vertices which induce connected subgraphs of the graph $G = (N, E)$. If G is a tree, then Owen [14, Theorems 6 and 7] gave the following formula for computing the dividends in the game $v^{\mathscr{L}}$,

$$\Delta_S(v^{\mathscr{L}}) = \sum_{\{T \subseteq N \,|\, ex(S) \subseteq T \subseteq S\}} \Delta_T(v).$$

Next, this formula is extended to the case when the graph is a connected block graph. Indeed, the formula holds in every partition convex geometry and can be showed by means of the Minkowski-Krein-Milman property.

Proposition 2. *Let $(N, \mathscr{L})$ be a partition convex geometry and let (N, v) be a game. The dividends of $S \in \mathscr{L}$ in the restricted game $v^{\mathscr{L}}$ are*

$$\Delta_S(v^{\mathscr{L}}) = \sum_{\{T \subseteq N \,|\, \bar{T} = S\}} \Delta_T(v) = \sum_{\{T \subseteq N \,|\, ex(S) \subseteq T \subseteq S\}} \Delta_T(v).$$

Proof: By formula (3), $v^{\mathscr{L}} = \sum_{T \subseteq N} \Delta_T(v) u_T^{\mathscr{L}}$ and by Theorem 1, $v^{\mathscr{L}} = \sum_{S \in \mathscr{L}} \Delta_S(v^{\mathscr{L}}) u_S$. We claim that $u_T^{\mathscr{L}} = u_{\bar{T}}$ for every nonempty coalition $T \subseteq N$. To verify this claim, consider the following equivalent conditions for all $S \subseteq N$:

$$u_T^{\mathscr{L}}(S) = 1 \Leftrightarrow \exists C \in \mathscr{L} \text{ such that } T \subseteq C \subseteq S \Leftrightarrow \bar{T} \subseteq S \Leftrightarrow u_{\bar{T}}(S) = 1.$$

Then, the coefficients satisfy

$$\Delta_S(v^{\mathscr{L}}) = \sum_{\{T \subseteq N | \bar{T} = S\}} \Delta_T(v).$$

Next, we show that for all $S \in \mathscr{L}$,

$$\{T \subseteq N | \bar{T} = S\} = \{T \subseteq N | ex(S) \subseteq T \subseteq S\}.$$

Let $T \subseteq N$ be such that $\overline{T} = S$. Then, $T \subseteq \overline{T} = S$, and $S \in \mathscr{L}$. But since every convex set is the closure of its extreme points, the set $ex(S)$ is a minimal subset of S such that $\overline{ex(S)} = S$, hence $ex(S) \subseteq T \subseteq S$. Conversely, if $ex(S) \subseteq T \subseteq S$, we obtain $S = \overline{ex(S)} \subseteq \overline{T} \subseteq \overline{S} = S$. □

The relation between the dividends of Harsanyi in the restricted game $v^{\mathscr{L}}$ and the worths in the original game v is given by the next result.

Proposition 3. *Let $(N, \mathscr{L})$ be a partition convex geometry and let $v \in \Gamma^N$. If $v^{\mathscr{L}}$ is the restricted game associated to v then*

$$\Delta_S(v^{\mathscr{L}}) = \sum_{T \in [S^-, S]} (-1)^{|S|-|T|} v(T), \quad \textit{where } S^- = S \backslash ex(S).$$

Proof: It follows from Theorem 1 that

$$v(S) = v^{\mathscr{L}}(S) = \sum_{T \in \mathscr{L}} \Delta_T(v^{\mathscr{L}}) u_T(S) = \sum_{\{T \in \mathscr{L} | T \subseteq S\}} \Delta_T(v^{\mathscr{L}}), \quad \forall S \in \mathscr{L}.$$

The partition convex geometry is a lattice and its Möbius function is computed in [4, Th. 4.3]:

$$\mu(T, S) = \begin{cases} (-1)^{|S|-|T|}, & \text{if } S \backslash T \subseteq ex(S) \\ 0, & \text{otherwise.} \end{cases}$$

Then, the Möbius inversion formula of $\mathscr{L}$ implies (see [18, p. 116]) that

$$\begin{aligned} \Delta_S(v^{\mathscr{L}}) &= \sum_{\{T \in \mathscr{L} | T \subseteq S\}} v(T) \mu(T, S) \\ &= \sum_{\{T \in \mathscr{L} | S \backslash T \subseteq ex(S)\}} (-1)^{|S|-|T|} v(T). \end{aligned}$$

We know that $\{T \in \mathscr{L} | S \backslash T \subseteq ex(S)\} = [S^-, S]$ and so, we obtain the formula for the dividends. □

5. The Shapley and Banzhaf values

Let $(N, \mathscr{L})$ be a partition convex geometry and let (N, v) be a game. The *Shapley value* for the player i in the restricted game $v^{\mathscr{L}}$ is given by $\Phi_i(N, v^{\mathscr{L}})$, for all $i \in N$. The *Banzhaf value* for the player i in the game $v^{\mathscr{L}}$ is given by $\beta'_i(N, v^{\mathscr{L}})$, for all $i \in N$. If G is a connected block graph, then the Shapley value is the Myerson value. In terms of dividends [14, p. 212], we have

$$\Phi_i(N, v^{\mathscr{L}}) = \sum_{\{S \subseteq N | i \in S\}} \frac{\Delta_S(v^{\mathscr{L}})}{|S|}, \quad \beta'_i(N, v^{\mathscr{L}}) = \sum_{\{S \subseteq N | i \in S\}} \frac{\Delta_S(v^{\mathscr{L}})}{2^{|S \backslash i|}}. \tag{5}$$

Edelman and Jamison [4, Th. 4.2] proved that if $(N, \mathscr{L})$ is a convex geometry and $S \in \mathscr{L}$, then the interval $[S^-, S] = \{C \in \mathscr{L} | S^- \subseteq C \subseteq S\}$ is a Boolean algebra, where $S^- = S \backslash ex(S)$. Then, $[S^-, S]$ is isomorphic to $2^{ex(S)}$. Now, the interval $[T, T^+]$ is considered, where $T \in \mathscr{L}$ and $T^+ = \{i \in N | T \cup i \in \mathscr{L}\}$.

Proposition 4. *Let $(N, \mathscr{L})$ be a partition convex geometry. Then we have:*

(a) *If $T \in \mathscr{L}$ and $T \neq \emptyset$, then $[T, T^+]$ is a Boolean algebra isomorphic to $2^{T^+ \backslash T}$.*
(b) *If $T = \emptyset$, then $[T, T^+] = \mathscr{L}$.*

Proof:
(a) Since the interval $[T, T^+] = \{S \in \mathscr{L} | T \subseteq S \subseteq T^+\}$ is isomorphic to subsets of

$$T^+ \backslash T = \{j \in N \backslash T | T \cup j \in \mathscr{L}\},$$

the result is obtained.
(b) Property (P1) implies that $\{i\} \in \mathscr{L}$ for all $i \in N$. If $T = \emptyset$, then $T^+ = N$. $\square$

In the next theorem two *explicit formulas*, in terms of v, for the Shapley and Banzhaf values of the players in the restricted game $v^{\mathscr{L}}$, are proved. We need the following lemma.

Lemma 1. *The $(N, \mathscr{L})$ be a partition convex geometry and let $T \in \mathscr{L}, T \neq \emptyset$. Then,*

$$\{S \in \mathscr{L} | T \in [S^-, S]\} = [T, T^+].$$

Proof: We first show that if $S \in \mathscr{L}$ and $S^- \subseteq T \subseteq S$, then $S \in [T, T^+]$. Since $T \subseteq S$ it is sufficient to prove that $S \backslash T \subseteq T^+ \backslash T = \{j \in N \backslash T | T \cup j \in \mathscr{L}\}$.

For any $j \in S \backslash T, T \cup j \in [S^-, S]$ and we know that the interval is a Boolean algebra. This implies that $T \cup j \in \mathscr{L}$.

Conversely, suppose $S \in [T, T^+]$. Then by Proposition 4(a) we have $S \in \mathscr{L}$. We shall show that $S \backslash T \subseteq ex(S)$, i.e., $S^- \subseteq T \subseteq S$. Since $[T, T^+]$ is a Boolean algebra, we have that $j \in S \backslash T$ implies $S \backslash j \in [T, T^+]$ and hence $S \backslash j \in \mathscr{L}$. Therefore $j \in ex(S)$. $\square$

Theorem 3. *Let $(N, \mathscr{L})$ be a partition convex geometry and let (N, v) be a game. If it is considered the following collections,*

$$\mathscr{L}_i = \{T \in \mathscr{L} | i \in T\},$$

$$\mathscr{L}_i^+ = \{T \in \mathscr{L} | i \in ex(T), (T \backslash i)^+ = T^+\},$$

$$\mathscr{L}_i^\star = \{T \in \mathscr{L} | i \notin T, T \cup i \in \mathscr{L}, T^+ \neq (T \cup i)^+\},$$

for all $i \in N$, then:

(a) The Shapley value for the player i in the restricted game $v^{\mathscr{L}}$ satisfies

$$\Phi_i(N, v^{\mathscr{L}}) = \sum_{T \in \mathscr{L}_i^+} \frac{(t-1)!(t^+ - t)!}{t^+!}[v(T) - v(T \setminus i)]$$

$$+ \sum_{T \in \mathscr{L}_i \setminus \mathscr{L}_i^+} \frac{(t-1)!(t^+ - t)!}{t^+!} v(T) - \sum_{T \in \mathscr{L}_i^\star} \frac{(t)!(t^+ - t - 1)!}{t^+!} v(T).$$

(b) The Banzhaf value for the player i in the restricted game $v^{\mathscr{L}}$ satisfies

$$\beta_i'(N, v^{\mathscr{L}}) = \sum_{T \in \mathscr{L}_i^+} \frac{1}{2^{t^+-1}}[v(T) - v(T \setminus i)]$$

$$+ \sum_{T \in \mathscr{L}_i \setminus \mathscr{L}_i^+} \frac{1}{2^{t^+-1}} v(T) - \sum_{T \in \mathscr{L}_i^\star} \frac{1}{2^{t^+-1}} v(T),$$

where $t = |T|$, and $t^+ = |T^+|$.

Proof: (a) By Theorem 1, we know that $\Delta_S(v^{\mathscr{L}}) = 0$ unless $S \in \mathscr{L}$. We use the formula (5) and Proposition 3 for computing

$$\Phi_i(N, v^{\mathscr{L}}) = \sum_{\{S \in \mathscr{L} | i \in S\}} \frac{\Delta_S(v^{\mathscr{L}})}{|S|} = \sum_{\{S \in \mathscr{L} | i \in S\}} \frac{1}{|S|} \left[\sum_{T \in [S^-, S]} (-1)^{|S|-|T|} v(T) \right].$$

Reversing the order of summation, we obtain

$$\Phi_i(N, v^{\mathscr{L}}) = \sum_{T \in \mathscr{L}} \left[\sum_{\{S \in \mathscr{L} | i \in S, T \in [S^-, S]\}} \frac{(-1)^{|S|-|T|}}{|S|} \right] v(T) = \sum_{T \in \mathscr{L}} c_i(T) v(T).$$

We apply Lemma 1 to the term in brackets, and denote $s = |S|$ and $t = |T|$. Thus

$$c_i(T) = \sum_{\{S \in [T, T^+] | i \in S\}} \frac{(-1)^{s-t}}{s} = \sum_{\{S \in \mathscr{L} | T \cup i \subseteq S \subseteq T^+\}} \frac{(-1)^{s-t}}{s}.$$

First, the case $i \in T$ is considered. The interval $[T, T^+]$ is a Boolean algebra, hence the summation index is $\{S \in 2^N | T \subseteq S \subseteq T^+\}$. We consider $S = T \cup R$, where $R = S \setminus T$ and $r = |R|$. Then,

$$c_i(T) = \sum_{R \subseteq T^+ \setminus T} \frac{(-1)^r}{t + r} = \sum_{r=0}^{t^+ - t} \binom{t^+ - t}{r} \frac{(-1)^r}{t + r}$$

$$= \sum_{r=0}^{t^+ - t} \binom{t^+ - t}{r} (-1)^r \int_0^1 x^{t+r-1} \, dx$$

$$= \int_0^1 x^{t-1} \sum_{r=0}^{t^+-t} \binom{t^+ - t}{r} (-x)^r \, dx$$

$$= \int_0^1 x^{t-1} (1-x)^{t^+-t} \, dx$$

$$= \frac{(t-1)!(t^+ - t)!}{t^+!}.$$

Next, assume that $i \notin T$, hence the index is $\{S \in \mathscr{L} | T \cup i \subseteq S \subseteq T^+\}$. Then $i \in T^+$ and $T^+ = T \cup \{j \notin T | T \cup j \in \mathscr{L}\}$ implies that $T \cup i \in \mathscr{L}$. Now, the previous result yields ($[T \cup i, T^+]$ is a Boolean algebra):

$$c_i(T) = - \sum_{\{S \in 2^N | T \cup i \subseteq S \subseteq T^+\}} \frac{(-1)^{s-(t+1)}}{s} = - \frac{t!(t^+ - t - 1)!}{t^+!}.$$

Inserting the coefficients, we have

$$\Phi_i(N, v^{\mathscr{L}}) = \sum_{T \in \mathscr{L}_i} \frac{(t-1)!(t^+ - t)!}{t^+!} v(T) - \sum_{\{T \in \mathscr{L} | i \notin T, T \cup i \in \mathscr{L}\}} \frac{(t)!(t^+ - t - 1)!}{t^+!} v(T). \quad (6)$$

For any $T \in \mathscr{L}_i$, if $i \in ex(T)$ and $(T \backslash i)^+ = T^+$, then $T \in \mathscr{L}_i^+$, hence $T \backslash i \in \mathscr{L}$ and $c_i(T \backslash i) = -c_i(T)$. Consequently, its contribution to the sum is $c_i(T)[v(T) - v(T \backslash i)]$. If $T \in \mathscr{L}_i \backslash \mathscr{L}_i^+$, then its term of the sum is $c_i(T)v(T)$.

Finally, for any $T \in \mathscr{L}$ with $i \notin T$ and $T \cup i \in \mathscr{L}$, such that $T^+ \neq (T \cup i)^+$, i.e., $T \in \mathscr{L}_i^\star$, the coefficients $c_i(T)$ and $-c_i(T \cup i)$ are differents. Therefore, its contribution is $c_i(T)v(T)$ (where $i \notin T$ implies $c_i(T) < 0$).

(b) The proof of the formula of the Banzhaf value is similar to the proof of (a). The only difference is that the coefficients are:

$$c_i(T) = \sum_{r=0}^{t^+-t} \binom{t^+ - t}{r} (-1)^r \left(\frac{1}{2}\right)^{t+r-1} = \left(\frac{1}{2}\right)^{t^+-1}, \quad \text{if } i \in T,$$

$$c_i(T) = -\left(\frac{1}{2}\right)^{t^+-1}, \quad \text{if } i \notin T \text{ and } T \cup i \in \mathscr{L}. \qquad \square$$

Notice that if $\mathscr{L} = 2^N$, then $ex(T) = T$, and $T^+ = N$ for every $T \in \mathscr{L}$. Thus, the formulas of Theorem 3 are equals to the Shapley (1) and Banzhaf (2) values. Moreover, the equation (6) is equal to the equation of Shapley (see reprint in [17, p. 35]). The formulas for computing these values can be further simplified when the player is a extreme point of N. Before doing so, we will need a lemma.

Lemma 2. *Let $(N, \mathscr{L})$ be a partition convex geometry. If $i \in ex(N)$, then we obtain*

$$\{T \in \mathscr{L}_i \big| |T| \geq 2\} = \{T \in \mathscr{L}_i^+ \big| |T| \geq 2\}.$$

Proof: If $T \in \mathscr{L}_i$, then $T \backslash i = T \cap (N \backslash i) \in \mathscr{L}$ and so $i \in ex(T)$. For all $T \in \mathscr{L}_i$ with $|T| \geq 2$, note that $T \in \mathscr{L}_i^+ \Leftrightarrow (T \backslash i)^+ = T^+$. We prove that these sets are equals.

First, if $j \in T^+$, with $j \notin T$ then $(T \backslash i) \cup j = (N \backslash i) \cap (T \cup j) \in \mathscr{L}$. Hence, we stated that $j \in (T \backslash i)^+$.

Conversely, take $j \in (T \backslash i)^+$. If $j = i$ we obtain the result and if $j \neq i$, observe that $((T \backslash i) \cup j) \cap T = T \backslash i \neq \emptyset$ because $|T| \geq 2$. Then, by Proposition 1, $(T \backslash i) \cup j \cup T = T \cup j \in \mathscr{L}$, and so $j \in T^+$. □

Theorem 4. *Let $(N, \mathscr{L})$ be a partition convex geometry and let (N, v) be a game such that $v(\{i\}) = 0$ for all $i \in N$. If $i \in ex(N)$, then the values for the player i in $v^{\mathscr{L}}$ satisfy*

$$\Phi_i(N, v^{\mathscr{L}}) = \sum_{T \in \mathscr{L}} \frac{(t-1)!(t^+ - t)!}{t^+!} [v(T) - v(T \backslash i)],$$

$$\beta_i'(N, v^{\mathscr{L}}) = \sum_{T \in \mathscr{L}} \frac{1}{2^{t^+ - 1}} [v(T) - v(T \backslash i)],$$

where $t = |T|$ and $t^+ = |T^+|$.

Proof: By Lemma 2, if $T \neq \emptyset, i \notin T$, and $T \cup i \in \mathscr{L}$ then $(T \cup i)^+ = T^+$. Therefore, $\mathscr{L}_i^\star = \{T \in \mathscr{L} | i \notin T, T \cup i \in \mathscr{L}, (T \cup i)^+ \neq T^+\} = \{\emptyset\}$. The index of $c_i(T)[v(T) - v(T \backslash i)]$ in Theorem 3 is $\{T \in \mathscr{L}_i | |T| \geq 2\}$, and $[v(T) - v(T \backslash i)] = 0$, if $i \notin T$ or $|T| \leq 1$. □

The explicit formula for the potential of $v^{\mathscr{L}}$ can be obtained by a similar method to the one that is used in Theorem 3.

Theorem 5. *Let $(N, \mathscr{L})$ be a partition convex geometry and let (N, v) be a game. Then the potential of $v^{\mathscr{L}}$ satisfies*

$$P(N, v^{\mathscr{L}}) = \sum_{T \in \mathscr{L}} \frac{(t-1)!(t^+ - t)!}{t^+!} v(T), \quad \textit{where } t = |T|,\ t^+ = |T^+|.$$

Example 7: Let (N, v) be the *four-person apex* game, that is, the simple game with the winning coalitions,

$$\mathscr{W} = \{\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},N\}.$$

The Shapley and Banzhaf values are [15, p. 143]:

$$\Phi(N, v) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right), \quad \beta'(N, v) = \left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

The cooperation structure defined in Example 6 is considered. Thus, $(N, \leq)$ is a chain with $1 < 2 < 3 < 4$ and the partition convex geometry is

$$\mathscr{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, N\}.$$

The collection of coalitions which are both winning and convex is

$$\mathscr{W} \cap \mathscr{L} = \{\{1,2\}, \{1,2,3\}, \{2,3,4\}, N\}.$$

Theorem 4 is used to compute the values in $v^{\mathscr{L}}$ of the players 1 and 4,

$$\Phi_1(N, v^{\mathscr{L}}) = \frac{1}{3!}[v(12) - v(2)] + \frac{2}{4!}[v(123) - v(23)] = \frac{1}{4},$$

$$\Phi_4(N, v^{\mathscr{L}}) = \frac{2}{4!}[v(234) - v(23)] = \frac{1}{12},$$

$$\beta'_1(N, v^{\mathscr{L}}) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 = \frac{3}{8}, \quad \beta'_4(N, v^{\mathscr{L}}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

To calculate the values in $v^{\mathscr{L}}$ of the players 2 and 3, we use Theorem 3. If $i = 2$, then $\mathscr{W} \cap \mathscr{L}_2^+ = \emptyset$, $\mathscr{W} \cap (\mathscr{L}_2 \backslash \mathscr{L}_2^+) = \mathscr{W} \cap \mathscr{L}$ and $\mathscr{W} \cap \mathscr{L}_2^\star = \emptyset$. Now, it is followed,

$$\Phi_2(N, v^{\mathscr{L}}) = \frac{1}{3!}[v(12)] + \frac{2}{4!}[v(123) + v(234)] + \frac{3!}{4!}\, v(N) = \frac{7}{12},$$

$$\beta'_2(N, v^{\mathscr{L}}) = \left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{5}{8}.$$

For $i = 3$, $\mathscr{W} \cap \mathscr{L}_3^+ = \emptyset$, $\mathscr{W} \cap (\mathscr{L}_3 \backslash \mathscr{L}_3^+) = \{\{1,2,3\}, \{2,3,4\}, N\}$, $\mathscr{W} \cap \mathscr{L}_3^\star = \{\{1,2\}\}$. Hence,

$$\Phi_3(N, v^{\mathscr{L}}) = \frac{2}{4!}[v(123) + v(234)] + \frac{3!}{4!}\, v(N) - \frac{2}{3!}\, v(12) = \frac{1}{12},$$

$$\beta'_3(N, v^{\mathscr{L}}) = \frac{1}{8}.$$

Then, the Shapley and Banzhaf values in the restricted game $v^{\mathscr{L}}$ are

$$\Phi(N, v^{\mathscr{L}}) = \left(\frac{1}{4}, \frac{7}{12}, \frac{1}{12}, \frac{1}{12}\right), \quad \beta'(N, v^{\mathscr{L}}) = \left(\frac{3}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{8}\right).$$

6. Owen multilinear extension

The *multilinear extension* (MLE) of the game (N, v) is the function of n real variables (see Owen [15]),

$$f(v)[q_1, \ldots, q_n] = \sum_{S \subseteq N} \prod_{j \in S} q_j \Delta_S(v),$$

where $\Delta_S(v)$ is the dividend of S in the game (N, v). Owen showed that

$$\Phi_i(N, v) = \int_0^1 \frac{\partial f(v)}{\partial q_i}[t, \ldots, t]\,dt, \quad \beta_i'(N, v) = \frac{\partial f(v)}{\partial q_i}\left[\frac{1}{2}, \ldots, \frac{1}{2}\right].$$

Proposition 5. *Let $(N, \mathscr{L})$ be a partition convex geometry and let (N, v) be a game. Then, the MLE of $v^{\mathscr{L}}$ is given by*

$$f(v^{\mathscr{L}})[q_1, \ldots, q_n] = \sum_{S \in \mathscr{L}} \prod_{j \in S} q_j \left[\sum_{T \in [S^-, S]} (-1)^{|S|-|T|} v(T)\right].$$

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