EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

European Journal of Operational Research 119 (1999) 365–372

www.elsevier.com/locate/orms

The core of games on convex geometries

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Abstract

A game on a convex geometry is a real-valued function defined on the family $\mathscr{L}$ of the closed sets of a closure operator which satisfies the finite Minkowski–Krein–Milman property. If $\mathscr{L}$ is the Boolean algebra 2^N then we obtain a n-person cooperative game. We will introduce convex and quasi-convex games on convex geometries and we will study some properties of the core and the Weber set of these games.

Keywords: Core; Weber set; Convex geometry

1. Introduction

A *cooperative game* is a function v that assigns to each *coalition* $S \subseteq N$ of a finite set N of *players* a real number $v(S)$ with $v(\emptyset) = 0$. Assuming that the coalition N of all players will form, a solution concept will prescribe a distribution of the worth $v(N)$ among the players. Doubtless, the most attractive solution concept is the *core* of the game. The *core* of a game $v : 2^N \to \mathbb{R}$, is the set of the vectors $x \in \mathbb{R}^n$ with $\sum_{i \in N} x_i = v(N)$ and $\sum_{i \in S} x_i \geqslant v(S)$, for all the coalitions $S \in 2^N$. Notice that no coalition $S \subseteq N$ should be able to improve upon x.

In this paper, we develop a model of cooperative games in which only certain coalitions are allowed to form. We will study the structure of such allowable coalitions using the theory of *convex geometries*, a notion developed to combinatorially abstract geometric convexity. In this context, the core is defined for the previous relations, but only for feasible coalitions. There have been previous models developed for Myerson [11], Faigle [5] and Faigle and Kern [6,7].

We begin by defining a convex geometry and describing some of their fundamental properties. Section 3 will define what we mean by the core for a game on a convex geometry. In Section 4 we introduce the Weber set as the convex hull of the marginal worth vectors. In the classical situation, a game is convex if and only if the Weber set coincides with the core of the game. For a game v on a convex geometry $\mathscr{L} \subseteq 2^N$, the inclusion $\mathrm{Core}(\mathscr{L}, v) \subseteq \mathrm{Weber}(\mathscr{L}, v)$ is not true. However, in the last section we show that for games on convex geometries the Weber set is contained in the core if and only if the game is quasi-convex. If the game is monotone then the result also holds for convex games.

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2. Games on convex geometries

In this section, we define the concept of *convex geometry* [4] and describe its fundamental properties. Let $N = \{1, 2, \ldots, n\}$ be a finite set and consider a family $\mathscr{L}$ of subsets of N with the properties

(1) $\emptyset \in \mathscr{L}$ and $N \in \mathscr{L}$,

(2) $A \in \mathscr{L}$ and $B \in \mathscr{L}$ implies that $A \cap B \in \mathscr{L}$.

The family $\mathscr{L}$ gives rise to the operator $- : 2^N \to 2^N$ defined by

$$A \mapsto \overline{A} := \bigcap \{C \in \mathscr{L} : A \subseteq C\}.$$

The operator – has the following properties: $A \subseteq \overline{A}$, $\overline{\overline{A}} = \overline{A}$ and $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$, for all $A, B \subseteq N$ with the additional condition that $\emptyset = \overline{\emptyset}$. That is, the operator – is a closure operator and $(N, -)$ is a closure space [1]. Conversely, every operator with the above conditions defines a family $\mathscr{L} \subseteq 2^N$ with properties (1) and (2) as the family of its closed sets $\mathscr{L} := \{A \subseteq N \colon A = \overline{A}\}$.

If $(N, -)$ is a closure space then $\mathscr{L} \subseteq 2^N$, ordered by inclusion, is a complete lattice in which meet and join operations are defined by

$$A \wedge B = A \cap B, \quad A \vee B = \overline{A \cup B},$$

for all $A, B \in \mathscr{L}$.

Throughout the remainder of this paper the closure space $(N, -)$ with the family $\mathscr{L}$ of its closed sets is identified.

Definition 1. The family of the closed sets $\mathscr{L}$ is a convex geometry if it satisfies the anti-exchange property:

$$\text{For every } A \subseteq N, \quad \text{if } i, j \notin \overline{A} \quad \text{and} \quad j \in \overline{A \cup i} \quad \text{then} \quad i \notin \overline{A \cup j}.$$

A set in a convex geometry $\mathscr{L}$ is called *convex*. For $A \subseteq N$ an element $a \in A$ is an *extreme point* of A if $a \notin \overline{A \setminus a}$. For a closed set $A \in \mathscr{L}$ this is equivalent to $A \setminus a \in \mathscr{L}$. Let $\mathrm{ex}(A)$ be the set of all extreme points of A. The convex geometries are the abstract closure spaces satisfying the finite Minkowski–Krein–Milman property: *Every closed set is the closure of its extreme points* [4]. The following result shows that convex geometries have some properties of Euclidean convexity.

Theorem 1. *Let* $- : 2^N \to 2^N$ *be a closure operator on N and let* $\mathscr{L}$ *be the family of its closed sets. Then the following statements are equivalent*:

(a) $\mathscr{L}$ *is a convex geometry.*

(b) *If* $A \neq N$ *is a closed set then* $A \cup \{i\}$ *is closed for some* $i \in N \setminus A$.

(c) *For every closed set* $A \subseteq N$, $A = \overline{\mathrm{ex}(A)}$.

Proof. See Edelman and Jamison ([4], Theorem 2.1). □

A *cooperative game* is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The players are the elements of N and the coalitions are the elements of the Boolean algebra 2^N.

Definition 2. A game on a convex geometry $\mathscr{L}$ is a function $v : \mathscr{L} \to \mathbb{R}$ such that $v(\emptyset) = 0$.

The coalitions are the convex sets of $\mathscr{L}$ and the players are the elements $i \in N$. Let $\Gamma(\mathscr{L})$ be the vector space over $\mathbb{R}$ of all games on the convex geometry $\mathscr{L} \subseteq 2^N$. A game on a convex geometry is called monotone or convex when $v : \mathscr{L} \to \mathbb{R}$ satisfies the corresponding property for the partial order and the join and meet operations.

Example 1. A *communication situation* is a triple (N, G, v), where (N, v) is a game and $G = (N, E)$ is a graph. This concept was first introduced in Myerson [11], and investigated in Borm et al. [3]. If $G = (N, E)$ is a connected block-graph ([9], p. 30), then the family of all coalitions of N that induce connected subgraphs

$$\mathscr{L} = \{S \subseteq N \colon (S, E(S)) \text{ is connected}\},$$

is a convex geometry ([4], Theorem 3.7).

Example 2. A subset S of a partially ordered set (poset) $(P, \leqslant)$ is *convex* whenever $a \in S$, $b \in S$ and $a \leqslant b$ imply $[a, b] \subseteq S$. The convex subsets of any poset P form a closure system $\mathrm{Co}(P)$. If P (or, equivalently $\mathrm{Co}(P)$) is finite, then each element is between a maximal and a minimal one. If $C \in \mathrm{Co}(P)$ then $\mathrm{ex}(C)$ is the union of the maximal and minimal elements of C. Moreover, $\mathrm{Co}(P)$ is a convex geometry ([2], Theorem 3).

Example 3. Let $(P, \leqslant)$ be a poset. For any $X \subseteq P$,

$$X \mapsto \overline{X} := \{y \in P: y \leqslant x \text{ for some } x \in X\}$$

defines a closure operator on P. Its closed sets are the *order ideals* (down sets) of P and we denote this lattice $J(P)$. Since the union and intersection of order ideals is again an order ideal, it follows that $J(P)$ is a sublattice of 2^P. Then $J(P)$ is a distributive lattice and so, $J(P)$ is a convex geometry closed under set-union and $\mathrm{ex}(S)$ is the set of all maximal points $\mathrm{Max}(S)$ of the subposet $S \in J(P)$. When P is finite, there is a 1–1 correspondence between antichains of P and order ideals. Then the games $(\mathscr{C}, v)$ and $(\mathscr{A}, c)$ of Faigle and Kern, where $\mathscr{C}$ is the family of down sets of P [6] and $\mathscr{A}$ is the set of antichains of a *hierarchy* [7], are games on distributive lattices.

3. The core of games on convex geometries

In the first part of this section we give some standard results of polyhedrons (see [13]). A set $P \subseteq \mathbb{R}^n$ is called a *polyhedron* if there exists a matrix A and a column vector b such that $P = \{x \in \mathbb{R}^n\colon Ax \leqslant b\}$. A set $P \subseteq \mathbb{R}^n$ is called a *polytope* if there exist $x_1, \ldots, x_t \in \mathbb{R}^n$ such that $P = \mathrm{conv}\{x_1, \ldots, x_t\}$. A set $P \subseteq \mathbb{R}^n$ is a polytope if and only if P is a bounded polyhedron ([13], p. 89). The polyhedron P is bounded if and only if $\{x \in \mathbb{R}^n\colon Ax \leqslant 0\} = \{0\}$ ([13], p. 100).

A *vertex* of P is an element of P which is not a convex combination of two other elements of P. If the polyhedron P has at least one vertex, then P is called *pointed*. The polyhedron P is pointed if and only if $\{x \in \mathbb{R}^n\colon Ax = 0\} = \{0\}$.

Now, we define the *core* of games with restricted cooperation.

Definition 3. Let $v \in \Gamma(\mathscr{L})$. The core of the game v is the set

$$\mathrm{Core}(\mathscr{L}, v) := \{x \in \mathbb{R}^n : x(N) = v(N),\ x(S) \geqslant v(S) \text{ for all } S \in \mathscr{L}\},$$

where for any $S \in \mathscr{L}$, $x(S) = \sum_{i \in S} x_i$, and $x(\emptyset) = 0$.

The following characterization for games on set systems with a non empty core is shown by Faigle [5]. The *indicator function* $\mathbf{1}_S : N \to \{0, 1\}$ for the subset $S \subseteq N$ is given by

$$\mathbf{1}_S(i) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2. *Let $\mathscr{L} \subseteq 2^N$ be a family such that $\emptyset, N \in \mathscr{L}$ and let $v : \mathscr{L} \to \mathbb{R}$ be a game. Then $\mathrm{Core}(\mathscr{L}, v) \neq \emptyset$ if and only if for all $S_1, \ldots, S_n \in \mathscr{L} \setminus \{\emptyset\}$ and $m \in \mathbb{N}$,*

$$\frac{1}{m}\sum_{i=1}^{n} \mathbf{1}_{S_i} = \mathbf{1}_N \text{ implies } \frac{1}{m}\sum_{i=1}^{n} v(S_i) \leqslant v(N).$$

Proof. See Theorem 4 of Faigle [5]. □

Furthermore, $\mathrm{Core}(\mathscr{L}, v) \neq \emptyset$ if and only if

$$\min\{\langle \mathbf{1}_N, x\rangle\colon \langle \mathbf{1}_S, x\rangle \geqslant v(S) \text{ for all } S \in \mathscr{L}\} \leqslant v(N).$$

The duality theorem of linear programming ([13], p. 90) implies the following result.

Theorem 3. *Let $\mathscr{L} \subseteq 2^N$ be a family such that $\emptyset, N \in \mathscr{L}$ and let $v : \mathscr{L} \to \mathbb{R}$ be a game. Then $\mathrm{Core}(\mathscr{L}, v) \neq \emptyset$ if and only if for y_S, $y_S \geqslant 0$, for all $S \in \mathscr{L} \setminus \{\emptyset\}$,*

$$\sum_{S \in \mathscr{L} \setminus \emptyset} y_S \mathbf{1}_S = \mathbf{1}_N \text{ implies } \sum_{S \in \mathscr{L} \setminus \emptyset} y_S v(S) \leqslant v(N).$$

Proposition 1. *The core of a game on a convex geometry is either empty or a polyhedron pointed.*

Proof. For all $i \in N$, we know that $i \in \mathrm{ex}(\overline{\{i\}})$. If $S = \overline{\{i\}}$ then $S \setminus i \in \mathscr{L}$. Since $x_i = x(S) - x(S \setminus i)$, we obtain

$$\{x \in \mathbb{R}^n : x(S) = 0, \text{ for all } S \in \mathscr{L}\} = \{0\}. \qquad \square$$

Definition 4. The positive core of the game $v \in \Gamma(\mathscr{L})$ is defined by

$$\mathrm{Core}^{+}(\mathscr{L},v) := \{x \in \mathrm{Core}(\mathscr{L},v) : x_i \geqslant 0 \text{ for all } i \in N\}.$$

Theorem 4. *Let v be a game on a convex geometry $\mathscr{L} \subseteq 2^N$ such that $\mathrm{Core}(\mathscr{L},v)$ is non-empty and $v(S) \geqslant 0$. Then, the following statements are equivalent*:

(a) *The core of the game v is a polytope.*

(b) *The atoms $\{i\} \in \mathscr{L}$, for all $i \in N$.*

(c) $\mathrm{Core}(\mathscr{L},v) = \mathrm{Core}^{+}(\mathscr{L},v)$.

Proof. (a) $\Rightarrow$ (b) If there exists $j \in N$ such that $\{j\} \notin \mathscr{L}$ we consider $\overline{\{j\}} \in \mathscr{L}$. Let $k \in \overline{\{j\}}$ with $k \neq j$, and we define the vector $x \in \mathbb{R}^n$,

$$x_i = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

This vector $x \neq 0$ satisfies $x(N) = 0$, and $x(S) \geqslant 0$, for all $S \in \mathscr{L}$, because if $j \in S$ then $\overline{\{j\}} \subseteq S$, hence $k \in S$. This is a contradiction with statement (a).

(b) $\Rightarrow$ (a) If $\{i\} \in \mathscr{L}$ for all $i \in N$, then

$$\{x \in \mathbb{R}^n : x(N) = 0,\ x(S) \geqslant 0, \text{for all } S \in \mathscr{L}\} = \{0\}.$$

Therefore, the polyhedron $\mathrm{Core}(\mathscr{L},v)$ is bounded.

(b) $\Longleftrightarrow$ (c) This equivalence is shown by Faigle ([5], Theorem 9) for games on closure spaces. □

4. The Weber set of games on convex geometries

The classic result states that if $v : 2^N \to \mathbb{R}$, is a *convex* (*supermodular*) game then $\mathrm{Core}(v) = \mathrm{Weber}(v)$, where the *Weber set* is the convex hull of the marginal worth vectors for v [12]. In 1978, Weber (see [14]) has shown that any game satisfies $\mathrm{Core}(v) \subseteq \mathrm{Weber}(v)$. Ichiishi [10] proved that if $\mathrm{Weber}(v) \subseteq \mathrm{Core}(v)$ then v is a convex game. Thus, these results imply the following characterization of convex games:

$$v : 2^N \to \mathbb{R}, \text{ is a convex game if and only if } \mathrm{Core}(v) = \mathrm{Weber}(v).$$

We study these concepts for games on convex geometries. Edelman and Jamison defined a *compatible ordering* of a convex geometry $\mathscr{L} \subseteq 2^N$ as a total ordering of the elements of N, $i_1 < i_2 < \cdots < i_n$ such that the set

$$\{i_1, i_2, \ldots, i_k\} \in \mathscr{L} \quad \text{for all } 1 \leqslant k \leqslant n.$$

A compatible ordering of $\mathscr{L}$ corresponds exactly to a maximal chain in $\mathscr{L}$. We denote by $\mathscr{C}(\mathscr{L})$ the set of all the maximal chains of $\mathscr{L}$. Given $i \in N$, and a maximal chain C, let

$$C(i) := \{j \in N : j \leqslant i \text{ in the chain } C\}.$$

Definition 5. Let $v \in \Gamma(\mathscr{L})$ and $C \in \mathscr{C}(\mathscr{L})$. The marginal worth vector $a^C \in \mathbb{R}^n$ with respect to the chain C in the game v is given by

$$a_i^C := v(C(i)) - v(C(i) \setminus i), \quad \text{for all } i \in N.$$

The ith coordinate a_i^C represents the marginal contribution of player i to the coalition of his predecessors with respect to the chain C.

Proposition 2. *Let $v \in \Gamma(\mathscr{L})$ and $C \in \mathscr{C}(\mathscr{L})$. For all $S \in C$, we have*

$$\sum_{j \in S} a_j^C = v(S).$$

Proof. Let $v \in \Gamma(\mathscr{L})$ and $C \in \mathscr{C}(\mathscr{L})$. We denote by S_k the coalition of C such that $|S_k| = k$ for any $k \in N$. Put $S_0 = \emptyset$ and $S_k = \{i_1, i_2, \ldots, i_k\}$ for all $1 \leqslant k \leqslant n$. By definition, we have that $a_i^C = v(C(i)) - v(C(i) \setminus i)$ and therefore it follows that

$$\begin{aligned} \sum_{j \in S_k} a_j^C &= \sum_{j=1}^{k} a_{i_j}^C \\ &= \sum_{j=1}^{k} [v(C(i_j)) - v(C(i_j) \setminus i_j)] \\ &= \sum_{j=1}^{k} [v(S_j) - v(S_{j-1})] \\ &= v(S_k), \end{aligned}$$

for all $k \in N$. Note that $S_n = N$ and hence $\sum_{j \in N} a_j^C = v(N)$. □

Definition 6. The Weber set of a game $v \in \Gamma(\mathscr{L})$ is the convex hull of the marginal worth vectors,

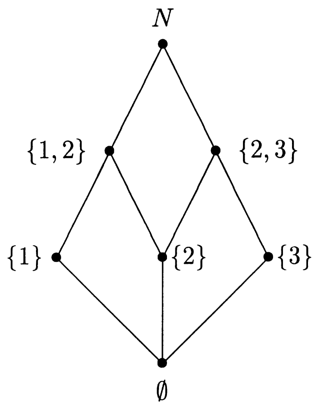
$$\text{Weber}(\mathscr{L}, v) := \text{conv}\left\{a^C : C \in \mathscr{C}(\mathscr{L})\right\}.$$

We obtain the relation between the core and the Weber set of a game on a convex geometry. If $\mathscr{L}$ is the Boolean algebra 2^N and v is an n-person cooperative game, then the core of v is contained in the Weber set. This result is due to Weber [14]. However, the inclusion $\text{Core}(\mathscr{L}, v) \subseteq \text{Weber}(\mathscr{L}, v)$ does not hold when v is a game on a convex geometry $\mathscr{L} \neq 2^N$.

In the following examples, we will use the convex geometry $\text{Co}(\mathbf{3}) \subseteq 2^N$, where $N = \{1,2,3\}$ and $\text{Co}(\mathbf{3}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$, is the collection of the convex subsets of the poset $\mathbf{3} = \{1 < 3\}$. There are four maximal chains in $\mathscr{L}$

$$C_1 : \emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\},$$
$$C_2 : \emptyset \subset \{2\} \subset \{1,2\} \subset \{1,2,3\},$$
$$C_3 : \emptyset \subset \{2\} \subset \{2,3\} \subset \{1,2,3\},$$
$$C_4 : \emptyset \subset \{3\} \subset \{2,3\} \subset \{1,2,3\}.$$

The Hasse diagram of the lattice Co(**3**) is



Example 4. Let $v : \text{Co}(\mathbf{3}) \to \mathbb{R}$ be a game, given by

$$v(1) = v(2) = v(3) = 0,$$
$$v(12) = v(23) = 2,$$
$$v(N) = 3.$$

The marginal worth vectors in the game v are

$$\begin{aligned} a^{C_1} &= (v(1) - v(\emptyset), v(12) - v(1), v(N) - v(12)) \\ &= (0,2,1), \\ a^{C_2} &= (v(12) - v(2), v(2) - v(\emptyset), v(N) - v(12)) \\ &= (2,0,1), \\ a^{C_3} &= (v(N) - v(23), v(2) - v(\emptyset), v(23) - v(2)) \\ &= (1,0,2), \\ a^{C_4} &= (v(N) - v(23), v(23) - v(3), v(3) - v(\emptyset)) \\ &= (1,2,0), \end{aligned}$$

and hence

$$\text{Weber}(\mathscr{L}, v) = \text{conv}\{(0,2,1), (2,0,1), (1,0,2), (1,2,0)\}.$$

On the other hand, the core of v is

$$\begin{aligned} \text{Core}(\mathscr{L}, v) &= \left\{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 3, x_3 \leqslant 1,\ x_1 \leqslant 1\right\} \\ &= \text{conv}\{(0,2,1), (1,2,0), (0,3,0), (1,1,1)\}. \end{aligned}$$

Then we have that $\text{Core}(\mathscr{L}, v) \nsubseteq \text{Weber}(\mathscr{L}, v)$ and $\text{Weber}(\mathscr{L}, v) \nsubseteq \text{Core}(\mathscr{L}, v)$.

Example 5. Let $v : \text{Co}(\mathbf{3}) \to \mathbb{R}$ be a game, given by

$$v(1) = v(2) = v(3) = 0,$$
$$v(12) = 1,$$
$$v(23) = 0,$$
$$v(N) = 3.$$

The marginal worth vectors in this game v are

$$a^{C_1} = (0,1,2),\ a^{C_2} = (1,0,2),\ a^{C_3} = (3,0,0),$$
$$a^{C_4} = (3,0,0),$$

and the Weber set is

$$\text{Weber}(\mathscr{L}, v) = \text{conv}\{(0,1,2), (1,0,2), (3,0,0)\}.$$

Now, we get that the core is the set

$$\begin{aligned} \text{Core}(\mathscr{L}, v) &= \left\{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 3,\ x_3 \leqslant 2, x_1 \leqslant 3\right\} \\ &= \text{conv}\{(0,1,2), (1,0,2), (3,0,0), (0,3,0)\}. \end{aligned}$$

In this example, each marginal worth vector coincides with some vertex of the core and hence $\mathrm{Weber}(\mathscr{L}, v) \subseteq \mathrm{Core}(\mathscr{L}, v)$. The following proposition provides a special type of vertices of the core.

Proposition 3. *Let $v \in \Gamma(\mathscr{L})$ and $C \in \mathscr{C}(\mathscr{L})$. If the vector $a^C \in \mathrm{Core}(\mathscr{L}, v)$, then a^C is a vertex of $\mathrm{Core}(\mathscr{L}, v)$.*

Proof. By Proposition 2, we have

$$\sum_{j\in S} a_j^C = v(S)$$

for all $S \in C$. Since each maximal chain in $\mathscr{L}$ has n non-empty coalitions, we get n equations and due to their linear independence, we can conclude that a^C is a vertex of the polyhedron $\mathrm{Core}(\mathscr{L}, v)$. □

5. Convex and quasi-convex games

Faigle and Kern [7] introduced the concept of a convex game for a distributive lattice $(\mathscr{A}, \vee, \wedge)$. In our model, a convex geometry $\mathscr{L}$ is a lattice with the join and meet operations

$$S \vee T = \overline{S \cup T}, \quad S \wedge T = S \cap T \quad \text{for all } S, T \in \mathscr{L}.$$

Definition 7. A game $v \in \Gamma(\mathscr{L})$ is said to be convex or supermodular if for all $S, T \in \mathscr{L}$,

$$v(S \vee T) + v(S \wedge T) \geqslant v(S) + v(T).$$

We introduce the concept of a quasi-convex game for studying the core of the game.

Definition 8. A game $v \in \Gamma(\mathscr{L})$ is quasi-convex if for all $S, T \in \mathscr{L}$ with $S \cup T \in \mathscr{L}$, we have

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T).$$

It is obvious that convex implies quasi-convex.

Remark 1. Note that if there is only a maximal chain in $\mathscr{L}$, all games defined on $\mathscr{L}$ are convex.

Proposition 4. *A game $v \in \Gamma(\mathscr{L})$ is quasi-convex if and only if for all $S, T \in \mathscr{L}$ such that $T \subset S$ and for all $i \in \mathrm{ex}(S) \cap T$, we have*

$$v(S) - v(S \setminus i) \geqslant v(T) - v(T \setminus i).$$

Proof. Let $S, T \in \mathscr{L}$ such that $T \subset S$ and let $i \in \mathrm{ex}(S) \cap T$. Consider $S' = S \setminus i$ and $T' = T$. We have

$$S' \cap T' = (S \setminus i) \cap T = T \setminus i \in \mathscr{L},$$
$$S' \cup T' = (S \setminus i) \cup T = S \in \mathscr{L}.$$

Applying the definition of quasi-convexity to S' and T', it follows that

$$v(S) + v(T \setminus i) \geqslant v(S \setminus i) + v(T).$$

Conversely, let $S, T \in \mathscr{L}$ such that $S \cup T \in \mathscr{L}$. If $T \subset S$ or $S \subset T$, then the equality is clear. Consider $S \cap T \neq S$ and $S \cap T \neq T$ and let $C \in \mathscr{C}(\mathscr{L})$ be a maximal chain such that $S \cap T$, T, $S \cup T \in C$. As $S \setminus T \neq \emptyset$, put $|S \setminus T| = k$ and write $S \setminus T = \{i_1, i_2, \ldots, i_k\}$ with $C(i_1) \subset C(i_2) \subset \cdots \subset C(i_k)$, i.e., the chain C is given by

$$\cdots \subset T \subset T \cup \{i_1\} \subset T \cup \{i_1, i_2\} \subset \cdots$$
$$\subset T \cup \{i_1, i_2, \ldots, i_k\} = T \cup S \subset \cdots$$

Let $R = S \cap T$ and denote $S_j = \{i_1, i_2, \ldots, i_j\}$ for all $1 \leqslant j \leqslant k$ with $S_0 = \emptyset$. Then we have that for all $1 \leqslant j \leqslant k$, $T \cup S_j \in \mathscr{L}$ and

$$R \cup S_j = (S \cap T) \cup S_j = (S \cup S_j) \cap (T \cup S_j)$$
$$= S \cap (T \cup S_j) \in \mathscr{L}.$$

Now, considering the hypothesis, we get

$$v(R \cup S_j) - v(R \cup S_{j-1}) \leqslant v(T \cup S_j) - v(T \cup S_{j-1}),$$

since $R \cup S_j \subset T \cup S_j$ and

$$i_j \in \mathrm{ex}(R \cup S_j) \cap (T \cup S_j),$$

and now it follows that

$$\begin{aligned} v(S) - v(S \cap T) &= v(R \cup S_k) - v(R) \\ &= \sum_{j=1}^{k} \left[v(R \cup S_j) - v(R \cup S_{j-1}) \right] \\ &\leqslant \sum_{j=1}^{k} \left[v(T \cup S_j) - v(T \cup S_{j-1}) \right] \\ &= v(T \cup S) - v(T). \qquad \square \end{aligned}$$

Theorem 5. *A game v on a convex geometry $\mathscr{L}$ is quasi-convex if and only if* $a^C \in \mathrm{Core}(\mathscr{L}, v)$, *for all* $C \in \mathscr{C}(\mathscr{L})$.

Proof. ($\Rightarrow$) Let C be a maximal chain in $\mathscr{L}$. Proposition 2 implies that $\sum_{j\in S} a_j^C = v(S)$ for all $S \in C$. It remains to prove that $\sum_{j\in S} a_j^C \geqslant v(S)$ for all $S \notin C$. Let $S \in \mathscr{L}$ such that $S \notin C$ and put $|S| = s \geqslant 1$. Write $S = \{i_1, i_2, \ldots, i_s\}$, where $C(i_1) \subset C(i_2) \subset \cdots \subset C(i_s)$.

Denote by $S_j = \{i_1, i_2, \ldots, i_j\}$ for all $1 \leqslant j \leqslant s$ and $S_0 = \emptyset$. For all $1 \leqslant j \leqslant s$, we have $S_j = S \cap C(i_j) \in \mathscr{L}$ and also $i_j \in \mathrm{ex}(C(i_j))$. Proposition 4 implies that for all $1 \leqslant j \leqslant s$, we have

$$v(C(i_j)) - v(C(i_j) \setminus i_j) \geqslant v(S_j) - v(S_{j-1}),$$

and hence

$$\begin{aligned}\sum_{j\in S} a_j^C &= \sum_{j=1}^{s} a_{i_j}^C \\ &= \sum_{j=1}^{s} [v(C(i_j)) - v(C(i_j) \setminus i_j)] \\ &\geqslant \sum_{j=1}^{s} [v(S_j) - v(S_{j-1})] \\ &= v(S).\end{aligned}$$

($\Leftarrow$) For any $S, T \in \mathscr{L}$ with $S \cup T \in \mathscr{L}$, let $C \in \mathscr{C}(\mathscr{L})$ be a maximal chain containing $S \cap T$ and $S \cup T$. The marginal worth vector $a^C \in \mathbb{R}^n$, belongs to $\mathrm{Core}(\mathscr{L}, v)$, hence $a^C(S) \geqslant v(S)$ and $a^C(T) \geqslant v(T)$. By construction, $a^C(S \cup T) = v(S \cup T)$ and $a^C(S \cap T) = v(S \cap T)$. Therefore,

$$\begin{aligned}v(S) + v(T) &\leqslant a^C(S) + a^C(T) \\ &= a^C(S \cup T) + a^C(S \cap T) \\ &= v(S \cup T) + v(S \cap T). \qquad \square\end{aligned}$$

Corollary 1. *A game v on a convex geometry $\mathscr{L}$ is quasi-convex if and only if*

$$\mathrm{Weber}(\mathscr{L}, v) \subseteq \mathrm{Core}(\mathscr{L}, v).$$

Proof. It is a direct consequence of theorem since the core of a game $v \in \Gamma(\mathscr{L})$ is a convex set. $\square$

Remark 2. A similar characterization of submodular games on distributive lattices is given in Fujishige ([8], Theorem 3.18), who considered the greedy algorithm for obtaining the result.

Example 6. Let $v : \mathrm{Co}(\mathbf{3}) \to \mathbb{R}$ be a game, defined by

$$v(1) = 1,\ v(2) = -1,\ v(3) = 1,$$

$$v(12) = v(23) = 0,\ v(N) = 1.$$

This game is quasi-convex but it is not convex since for $S = \{1\}$ and $T = \{3\}$, we have $S \wedge T = \emptyset$, $S \vee T = \overline{\{1, 3\}} = N$ and

$$1 = v(N) + v(\emptyset) < v(1) + v(3) = 1 + 1 = 2.$$

The marginal worth vectors for v are:

$$a^{C_1} = a^{C_2} = a^{C_3} = a^{C_4} = (1, -1, 1),$$

and hence

$$\mathrm{Weber}(\mathscr{L}, v) = \mathrm{Core}(\mathscr{L}, v) = \{(1, -1, 1)\}.$$

In the above example, we obtain that the core and the Weber set are equals, for a non convex game. Then, the property $\mathrm{Weber}(\mathscr{L}, v) \subseteq \mathrm{Core}(\mathscr{L}, v)$ is not a characterization of the convex games on convex geometries.

Definition 9. A game $v \in \Gamma(\mathscr{L})$ is monotone if $S \subseteq T$ implies $v(S) \leqslant v(T)$.

Note that if v is monotone then

$$v(S) \geqslant v(\emptyset) = 0, \text{ for all } S \in \mathscr{L}.$$

Theorem 6. *Let $v \in \Gamma(\mathscr{L})$ be a monotone game. Then, the game v is convex if and only if* $\mathrm{Weber}(\mathscr{L}, v) \subseteq \mathrm{Core}(\mathscr{L}, v)$.

Proof. If v is convex then it is quasi-convex and corollary 1 of Theorem 5 implies the property. Conversely, for $S, T \in \mathscr{L}$, let C be a maximal chain containing to $S \vee T = \overline{S \cup T}$ and $S \wedge T = S \cap T$. We observe that $a^C(S) \geqslant v(S)$ and $a^C(T) \geqslant v(T)$, since the marginal worth vector $a^C \in \mathrm{Core}(\mathscr{L}, v)$. On the other hand, $a_i^C = v(C(i)) - v(C(i) \setminus i) \geqslant 0$ since v is monotonic and therefore $a^C \geqslant 0$.

By Proposition 2, $a^C(\overline{S \cup T}) = v(\overline{S \cup T})$ and $a^C(S \cap T) = v(S \cap T)$. We get the result from the following inequalities

$$\begin{aligned} v(S) + v(T) &\leqslant a^C(S) + a^C(T) \\ &= a^C(S \cup T) + a^C(S \cap T) \\ &\leqslant a^C(\overline{S \cup T}) + a^C(S \cap T) \\ &= v(\overline{S \cup T}) + v(S \cap T). \qquad \square \end{aligned}$$

References

[1] G. Birkhoff, Lattice Theory, vol. 15, Colloquium, AMS, Providence, RI, 1940.

[2] G. Birkhoff, M.K. Bennett, The convexity lattice of a poset, Order 2 (1985) 223–242.

[3] P. Borm, G. Owen, S. Tijs, On the position value for communication situations, SIAM Journal of Discrete Mathematics 5 (1992) 305–320.

[4] P.H. Edelman, R.E. Jamison, The theory of convex geometries, Geometriae Dedicata 19 (1985) 247–270.

[5] U. Faigle, Cores of games with restricted cooperation, ZOR-Methods and Models of Operations Research 33 (1989) 405–422.

[6] U. Faigle, W. Kern, The Shapley value for cooperative games under precedence constraints, International Journal of Game Theory 21 (1992) 249–266.

[7] U. Faigle, W. Kern, Partition games and the core of hierarchically convex cost games, Working Paper, University of Twente, The Netherlands, 1995.

[8] S. Fujishige, Submodular Functions and Optimization, North-Holland, Amsterdam, 1991.

[9] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.

[10] T. Ichiishi, Supermodularity: Applications to convex games and to the greedy algorithm for LP, Journal of Economic Theory 25 (1981) 283–286.

[11] R.B. Myerson, Graphs and cooperation in games, Mathematics of Operations Research 2 (1977) 225–229.

[12] L.S. Shapley, Cores of convex games, International Journal of Game Theory 1 (1971) 11–26.

[13] A. Schrijver, Theory of Linear and Integer Programming, Wiley, New York, 1986.

[14] R.J. Weber, Probabilistic values for games, in: A.E. Roth (Ed.), The Shapley Value, Cambridge University Press, Cambridge, 1988, pp. 101–119.