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Chapter 1

ALGORITHMS FOR COMPUTING THE MYERSON VALUE BY DIVIDENDS

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Abstract

The goal of this chapter is to compute the Myerson value of cooperative games restricted by a combinatorial structure. There have been previous models developed to study the problem of games with partial cooperation. Games restricted by a communication graph were introduced by Myerson [13] and Owen [14]. Another type of combinatorial structure introduced by Gilles, Owen and van den Brink [9] is equivalent to a subclass of antimatroids. Cooperative games in which the set of players is a partially ordered set, that is, games on distributive lattices, were investigated by Faigle and Kern [7]. We introduce a new combinatorial structure called *union stable system*, which is a generalization of the antimatroid structure and the system of connected subgraphs of a graph. We present new algorithmic procedures for computing values of games under union stable systems restrictions and we show that there exist problems with polynomial algorithm complexity.

1 Restricted games

A *cooperative game* (N,v) is a function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$. The *players* are the elements of the finite set $N = \{1,2,\dots,n\}$, and the *coalitions* are the subsets $S \subseteq N$. To every coalition S is associated its *characteristic vector* e^S, defined by $\left(e_i^S\right)_{i\in N} = 1$ if $i \in S$, and $\left(e_i^S\right)_{i\in N} = 0$ otherwise. This is the natural correspondence between 2^N and $\{0,1\}^n$. Through this identification of coalitions with their characteristic vectors, a cooperative game (N,v) is a *pseudo-Boolean* function $v : \{0,1\}^n \to \mathbb{R}$, with $v(\mathbf{0}) = 0$. Any pseudo-Boolean function has a unique expression as a multilinear polynomial in n discrete variables (see Hammer

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and Rudeanu [10]) given by

$$v(x) = \sum_{S \subseteq N} \left(d_v(S) \prod_{i \in S} x_i \right)$$

for all $x \in \{0,1\}^n$, where the coefficients $d_v(S) \in \mathbb{R}$ are the Harsanyi dividends [11] of the coalitions S in the cooperative game v.

In this chapter, we study cooperative games in which the cooperation among the players is partial. Several models of partial cooperation have been proposed, among which are those derived from *communication situations* as introduced by Myerson [13] and analyzed by Owen [14], and Borm, Owen and Tijs [4]. We will give special attention to the union stable systems and we will study the complexity of the algorithm that, by means of the Harsanyi dividends, allows us to compute the Shapley value of the restricted game, i.e., the *Myerson value*. Some results on the complexity of computing the Myerson value will be provided in section 2. In section 3, we consider *convex geometries* introduced by Edelman and Jamison [5]. This concept gives rise to a special type of union stable structure and generalizes those communication situations in which the graph that models the bilateral relations among players is a tree. Algaba, Bilbao, Borm and López [1, 2] consider a partial cooperation model based on the so-called *union stable systems* which is a generalization of the communication situations. Throughout this chapter N denotes a finite set, and we use $\mathcal{F} \subseteq 2^N$ to denote the set system $(N, \mathcal{F})$.

Definition 1 *A set system $\mathcal{F} \subseteq 2^N$ is called union stable if for all $A, B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$ it is satisfied that $A \cup B \in \mathcal{F}$.*

Example 1. A communication situation is a triple (N, v, E), where (N, v) is a game and $G = (N, E)$ is a graph. It is easy to see that the collection

$$\mathcal{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } G\},$$

is a union stable system.

A union stable system can not always be modelled by a communication graph. Let us consider $N = \{1,2,3,4\}$ and the collection

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3\}, \{2,3,4\}, N\}.$$

This set system is union stable, but does not coincide with the connected subgraph family of any graph.

Example 2. Permission structures were defined by Gilles, Owen and van den Brink [9]. They assume that players who participate in a cooperative game are restricted by a hierarchical organization in which there are players that need permission from certain other players before they are allowed to cooperate. Algaba et al. [2] showed that if the family $\mathcal{A}$ of subsets of N is derived from an disjunctive or conjunctive approach of an acyclic permission structure then $\mathcal{A}$ is a union stable system.

Example 3. Let $N=\{1,2,\ldots,n\}$ and consider the collection $\mathcal{F}_n$ of all the connected coalitions of the path $1-n$, that is,

$$\mathcal{F}_n=\{[i,j]:1\leq i\leq j\leq n\}\cup\{\emptyset\},$$

where $[i,j]=\{i,i+1,\ldots,j-1,j\}$. Then $\mathcal{F}_n$ is a union stable system which corresponds to a voting situation in a unidimensional policy order (see Edelman [6]).

Definition 2 *Consider $\mathcal{F}\subseteq 2^N$ and let $S\subseteq N$. A set $T\subseteq S$ is called a $\mathcal{F}$-component of S if it is satisfied that $T\in\mathcal{F}$ and there exists no $T'\in\mathcal{F}$ such that $T\subset T'\subseteq S$.*

The $\mathcal{F}$-components of S are the maximal coalitions that belong to $\mathcal{F}$ and are contained in S. We denote by $C_{\mathcal{F}}(S)$ the set of the $\mathcal{F}$-components of S. Observe that the set $C_{\mathcal{F}}(S)$ may be the empty set.

Proposition 3 *The set system $\mathcal{F}\subseteq 2^N$ is union stable if and only if for any $S\subseteq N$ such that $C_{\mathcal{F}}(S)\neq\emptyset$, the $\mathcal{F}$-components of S form a partition of a subset of S.*

Proof. Let $\mathcal{F}$ be a union stable system. Let $S^1\neq S^2$ be maximal feasible coalitions of S. If $S^1\cap S^2\neq\emptyset$, then $S^1\cup S^2\in\mathcal{F}$ since $\mathcal{F}$ is union stable and $S^1\cup S^2\subseteq S$. But S^1 and S^2 are $\mathcal{F}$-components of S and obtain a contradiction.

Conversely, assume for any S such that $C_{\mathcal{F}}(S)\neq\emptyset$, its $\mathcal{F}$-components form a partition of a subset of S. Suppose that $\mathcal{F}$ is not union stable, then there are $A,B\in\mathcal{F}$, with $A\cap B\neq\emptyset$ and $A\cup B\notin\mathcal{F}$. Hence, there must be an $\mathcal{F}$-component $C_1\in C_{\mathcal{F}}(A\cup B)$, with $A\subseteq C_1$ and an $\mathcal{F}$-component $C_2\in C_{\mathcal{F}}(A\cup B)$, with $B\subseteq C_2$ such that $C_1\neq C_2$. This contradicts the fact that the $\mathcal{F}$-components of $A\cup B$ are disjoint. □

Notice that if $\mathcal{F}$ is a union stable system such that $\{i\}\in\mathcal{F}$ for all $i\in N$, then the $\mathcal{F}$-components of S form a partition of S.

Definition 4 *Let (N,v) be a game and let $\mathcal{F}\subseteq 2^N$ be a union stable system. The $\mathcal{F}$-restricted game $v^{\mathcal{F}}:2^N\to\mathbb{R}$, is defined by*

$$v^{\mathcal{F}}(S):=\sum_{T\in C_{\mathcal{F}}(S)}v(T).$$

A *union stable structure* is a triple $(N,v,\mathcal{F})$ where (N,v) is a game and $\mathcal{F}\subseteq 2^N$ is a union stable system.

Definition 5 *The Myerson value of a union stable structure $(N,v,\mathcal{F})$ is given by the vector $\mu(N,v,\mathcal{F}):=\Phi\left(N,v^{\mathcal{F}}\right)$, where Φ is the Shapley value.*

By Γ^N we denote the set of all games (N,v). Given a union stable structure $(N,v,\mathcal{F})$, the set of the unanimity games $\{u_T:T\in\mathcal{F},\,T\neq\emptyset\}$ is a basis of the vector space $L_{\mathcal{F}}\left(\Gamma^N\right)$, where $L_{\mathcal{F}}:\Gamma^N\to\Gamma^N$, is defined by $L_{\mathcal{F}}(v)=v^{\mathcal{F}}$ (see Bilbao [3]). Then $v^{\mathcal{F}}$ can be expressed as a linear combination of the unanimity games corresponding to the feasible coalitions, that is, $v^{\mathcal{F}}=\sum_{T\in\mathcal{F}}d_{v^{\mathcal{F}}}(T)\,u_T$, where $d_{v^{\mathcal{F}}}(T)$ is the dividend of T in the game $v^{\mathcal{F}}$ and $d_{v^{\mathcal{F}}}(\emptyset)=$

0. The linearity of the Shapley value implies that the Myerson value satisfies, for every $i \in N$,

$$\mu_i(N, v, \mathcal{F}) = \sum_{\{S \in \mathcal{F} : i \in S\}} \frac{d_{v^{\mathcal{F}}}(S)}{|S|}.$$

Moreover, for every $S \in \mathcal{F}$, we have

$$v(S) = v^{\mathcal{F}}(S) = \sum_{\{T \in \mathcal{F} : T \subseteq S\}} d_{v^{\mathcal{F}}}(T).$$

From this expression we obtain the following recursive algorithm:

$$\begin{aligned} d_{v^{\mathcal{F}}}(\emptyset) &= 0, \\ d_{v^{\mathcal{F}}}(S) &= v(S) - \sum_{\{T \in \mathcal{F} : T \subset S\}} d_{v^{\mathcal{F}}}(T). \end{aligned}$$

The description of the above *dividend* algorithm is as follows:

Algorithm *dividend* $\left(N, v^{\mathcal{F}}\right)$

$d_{v^{\mathcal{F}}}(\emptyset) \leftarrow 0$

for i **from** 1 **to** n

  for j **from** 1 **to** $S(i)$

  $d_{v^{\mathcal{F}}}\left(S_i^j\right) \leftarrow v\left(S_i^j\right) - \sum_{\left\{T \in \mathcal{F} : T \subset S_i^j\right\}} d_{v^{\mathcal{F}}}(T)$

  end for

end for

where S_i^j is the j-th feasible coalition of size i and $S(i)$ is the number of feasible coalitions of cardinal i.

2 Algorithm complexity

The *time complexity* function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ of an algorithm A is the maximal number $f(n)$ of iterations of a universal Turing machine makes before halting, taken over all inputs of size n. We say that an algorithm has *space complexity* at most $f(n)$, if it can be computed by a Turing machine with space demand (cells and tapes) at most $f(n)$. An algorithm describes a sequence of operations to be performed on the given data; hence, the efficiency of an algorithm depends on the efficiency of these operations. For set of data elements, the efficiency of operations in turn depends on the data structure. We will work with a set $\mathcal{F}$ of data which is stored previously and hence we measure the execution time of the algorithms for computing dividends and values of restricted games.

Let f and g be functions from $\mathbb{Z}_+$ to $\mathbb{Z}_+$. We write $f(n) = \mathrm{O}(g(n))$, in words f *is of the order of* g, if there are positive integers c and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$. We write $f(n) = \Omega(g(n))$ if the opposite happens, that is, $g(n) = \mathrm{O}(f(n))$. If f and g have exactly the same rate of growth, then we write $f(n) = \Theta(g(n))$. For instance, if $p(n)$ is a polynomial of degree d, then $p(n) = \Theta(n^d)$. The above OΩΘ-notation was proposed by Knuth [12]. For a more detailed exposition, see the book of Gács and Lovász [8]. We have the following result for spatial and temporal complexity of the *dividend* algorithm.

Theorem 6 *Let $(N, v, \mathcal{F})$ be a union stable structure. To compute all dividends of the restricted game requires a space $\Omega(|\mathcal{F}|)$ and a time $O(3^n)$.*

Proof. First of all, notice that it is suffices to calculate dividends of the feasible coalitions and that the number of feasible coalitions of size i, denoted by $S(i)$, satisfies that $S(i) \leq \binom{n}{i}$ for $i = 1, \ldots, n$. The execution time of the *dividend* algorithm satisfies

$$\begin{aligned}
t(dividend) &= 1 + t(loop1) = 1 + \sum_{i=1}^{n} t(loop2) \\
&= 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)} t(assignment) = 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)} (1 + t(sum)) \\
&\leq 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)} \left(1 + \sum_{k=1}^{2^i-1} 1\right) = 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)} (1 + 2^i - 1) \\
&= 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)} 2^i = 1 + \sum_{i=1}^{n} S(i)\, 2^i \leq 1 + \sum_{i=1}^{n} \binom{n}{i} 2^i \\
&= \sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n.
\end{aligned}$$

Therefore, *dividend* has a time $O(3^n)$. On the other hand, if it is taken into account that the computation of the dividends is by an ascending process that must necessarily store the dividends of each one of the feasible coalitions, it is obtained that the required space is $\Omega(|\mathcal{F}|)$. □

Theorem 7 *Let $(N, v, \mathcal{F})$ be a union stable structure, v a zero-normalized game and let $p = |C_{\mathcal{F}}(N)|$. Then we have:*

a) To compute the Myerson value for a player i, by the dividend algorithm, requires a space $\Omega(|\mathcal{F}_M|)$ and a time $O\left(3^{|M|}\right)$, where $M \in C_{\mathcal{F}}(N)$ and $i \in M$.

b) To compute the Myerson value for all players, by the dividend algorithm, requires a space $\Omega(|\mathcal{F}|)$ and a time $O\left(p \cdot 3^{\max\{|M|: M \in C_{\mathcal{F}}(N)\}}\right)$.

Proof. *a)* If the coalition N is not feasible and v is zero-normalized, the calculation of the Myerson value, for player i, can be done through the maximal feasible coalition of N which the player i belongs to. This is, if $M \in C_{\mathcal{F}}(N)$ such that $i \in M$, then $(M, v_M, \mathcal{F}_M)$ is a union stable structure, where v_M is the restriction of v to M and $\mathcal{F}_M = \{F \in \mathcal{F} : F \subseteq M\}$, and it follows that $\mu_i(N, v, \mathcal{F}) = \mu_i(M, v_M, \mathcal{F}_M)$.

b) We will use two stages to compute the Myerson value of all players. In the first one, using dynamic programming, the dividends of all feasible coalitions are determined, for each $\mathcal{F}$-component of the coalition N by the following scheme: let $C_{\mathcal{F}}(N) = \{M_1, \ldots, M_p\}$, then

$d_{v^{\mathcal{F}}}(\emptyset) \leftarrow 0$

for j **from** 1 **to** p

$dividend\left(M_j, v^{\mathcal{F}}_{M_j}\right)$

$\{$To compute $d_{v^{\mathcal{F}}_{M_j}}(S)$, for all $S \in \mathcal{F}_{M_j}\}$

$\left\{d_{v^{\mathcal{F}}}(S) = d_{v^{\mathcal{F}}_{M_j}}(S)\right\}$

end for

In the second stage, the Myerson value is determined when all dividends are known. The algorithm is as follows

for j **from** 1 **to** p

for i **from** 1 **to** $\left|M_j\right|$

$\mu^j_i(N, v, \mathcal{F}) \leftarrow \sum_{\{S \in \mathcal{F}_{M_j} : i \in S\}} \dfrac{d_{v^{\mathcal{F}}_{M_j}}(S)}{|S|}$

$\left\{\mu^j_i(N, v, \mathcal{F}) = \mu_i\left(M_j, v_{M_j}, \mathcal{F}_{M_j}\right)\right\}$

end for

end for

Therefore, $t(Myerson) = f(n) + g(n)$, where $f(n)$ and $g(n)$ are the functions that respectively indicate the calculation time in the two established stages, i.e., the corresponding time to determinate the dividends and, once that dividends are known, the calculation time of the Myerson value for each one of the players. Indeed,

$$f(n) = t(loop) \leq p\,t\left(dividend\left(M, v^{\mathcal{F}}_M\right)\right) \leq p\,3^{|M|},$$

where $M \in C_{\mathcal{F}}(N)$ and satisfies that $|M| = \max\{\left|M_j\right|, 1 \leq j \leq p\}$. In the same way, we have

$$g(n) = t(loop1) = p\,t(loop2) \leq p|M|t(sum) \leq p|M|2^{|M|}.$$

Therefore,

$$\mathrm{O}(t(Myerson)) = \mathrm{O}(f(n) + g(n)) = \mathrm{O}\left(\max\left\{p\,3^{|M|}, p|M|2^{|M|}\right\}\right).$$

Finally, since

$$\lim_{|M| \to \infty} \frac{|M|2^{|M|}}{3^{|M|}} = 0,$$

we have that the computation of the Myerson value for all players requires a time $\mathrm{O}\big(\max\{p3^{|M|}, p|M|2^{|M|}\}\big) = \mathrm{O}\big(p3^{|M|}\big)$. Since all dividends have to be obtained the required space is $\Omega(|\mathcal{F}|)$. $\square$

Next, we analyze the complexity for the computation of the Myerson value in the communication situation in which the cooperation graph has a specific structure.

Consider (N,v,G), where G is a path $1-n$ with n vertices and (N,v) a game with transferable utility. If the vertices are enumerated from left to right, the set $\mathcal{F}$ of feasible coalitions is given by

$$\mathcal{F} = \{[i,j] : 1 \leq i \leq j \leq n\} \cup \{\emptyset\},$$

where $[i,j] = \{i, i+1, \ldots, j-1, j\}$.

Notice that except the empty set, the feasible coalitions can be arranged as entries on a matrix $A \in \mathcal{M}_n(\mathbb{R})$, upper triangular, such that the unitary coalitions are kept on the diagonal. On the row k, $k = 1, \ldots, n$, the coalitions of the set $\{[k,j] : k \leq j \leq n\}$ are arranged in an orderly way. The player k belongs to all coalitions of the submatrix (a_{ij}), for $i = 1, \ldots, k$ and $j = k, \ldots, n$. Hence, the total number of coalitions which contain player k is $F_k = k(n-k+1)$. It can also be observed that on the *i-th* superdiagonal would be all feasible coalitions with i players. On this, there is a number of elements equal to $(n-i+1)$ and, therefore, the number of feasible coalitions with i players is given by $S(i) = n-i+1$. Moreover, the total number of nonempty feasible coalitions is

$$|\mathcal{F}| = \sum_{i=1}^{n} S(i) = \sum_{i=1}^{n} (n-i+1) = \frac{n^2+n}{2}.$$

Example 4. Let us consider the path $1-n$ with $n=5$ vertices. Then the collection of feasible coalitions is

$$\mathcal{F} = \{[i,j] : 1 \leq i \leq j \leq 5\} \cup \{\emptyset\}.$$

The nonempty coalitions can be kept as entries of the following upper triangular matrix:

$$\begin{pmatrix} 1 & 12 & 123 & 1234 & 12345 \\ & 2 & 23 & 234 & 2345 \\ & & 3 & 34 & 345 \\ & & & 4 & 45 \\ & & & & 5 \end{pmatrix}$$

Note that the number of feasible coalitions with i players is

$$S(i) = 5 - i + 1 = 6 - i, \ \text{ for } \ i = 1, \ldots, 5.$$

In the same way, the number of coalitions which a player k belongs to is given by

$$F_k = k(5-k+1) = k(6-k), \ \text{ for } \ k = 1, \ldots, 5.$$

Theorem 8 *Let (N,v,G) be a communication situation, where G is a path $1-n$. To compute all dividends of the restricted game requires a space $\Omega(n^2)$ and a time $\Theta(n^4)$.*

Proof. For computing the dividends is used the *dividend* algorithm as above. Now then, in this particular case, the number of feasible coalitions of size i is given by $S(i) = n - i + 1$, and the total number of nonempty feasible coalitions contained in a feasible coalition T with $|T| = i$ is given by

$$\sum_{k=1}^{i}(i-k+1) = \frac{i^2+i}{2}.$$

Note that if the coalition T is excluded and the empty coalition is included, then the total number of feasible coalitions strictly contained in T is given by

$$C(i) = \frac{i^2+i}{2}.$$

Therefore, the execution time of the algorithm is

$$\begin{aligned}
t(dividend) &= 1 + t(loop1) = 1 + \sum_{i=1}^{n} t(loop2) \\
&= 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)} t(assignment) = 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)}(1 + t(sum)) \\
&= 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)}\left(1 + \sum_{k=1}^{C(i)} 1\right) = 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)}(1 + C(i)) \\
&= 1 + \sum_{i=1}^{n}\left(1 + \frac{i^2+i}{2}\right) S(i) \\
&= 1 + \sum_{i=1}^{n}\left(\frac{i^2+i+2}{2}\right)(n-i+1) \\
&= 1 + \frac{1}{2}\sum_{i=1}^{n}\left(i^2+i+2\right)(n-i+1) \\
&= \frac{n^4+6n^3+23n^2+18n+24}{24}.
\end{aligned}$$

Thus, the time $t(dividend) = \Theta(n^4)$. The recursive process requires to keep the dividends corresponding to all the feasible nonempty coalitions whose number is $\left(n^2+n\right)/2$. Therefore, the required space is $\Omega(n^2)$. $\square$

From the Myerson value expression in terms of the dividends of the restricted game, the following result is deduced, using dynamic programming.

Theorem 9 *Let (N, v, G) be a communication situation, where G is a path $1 - n$. To compute the Myerson value by the dividend algorithm requires a space $\Omega\left(n^2\right)$ and a time $\Theta\left(n^4\right)$.*

Proof. First of all, all dividends are calculated, using the *dividend* algorithm. Next, for computing the Myerson value of player k, it is necessary to consider that this player belongs to $F_k = k(n-k+1)$ feasible coalitions. Since

$$F_k = \frac{1}{4}(n+1)^2 - \left(\frac{1}{2}(n+1) - k\right)^2,$$

we obtain, for any value of k,

$$n \leq F_k \leq \frac{1}{4}(n+1)^2.$$

Then, we have that the required time to evaluate the sum in the expression of the Myerson value is $\mathrm{O}(n^2)$, for each player. Therefore, the complexity for computing the Myerson value is determined by the complexity for computing the dividends. So, the required time is $\Theta\left(n^4\right)$. The required space $\Omega(n^2)$ is an immediately consequence of Theorem 8. □

3 Restricted games by convex geometries

A *convex geometry* is a set system $\mathcal{F} \subseteq 2^N$ which satisfies the following properties:

(C1) $\emptyset \in \mathcal{F}$,

(C2) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$,

(C3) If $A \in \mathcal{F}$ and $A \neq N$, there exists $i \in N \setminus A$ such that $A \cup i \in \mathcal{F}$.

The elements of a convex geometry are called *convex sets*. Given a convex set $S \in \mathcal{F}$, an element $i \in S$ is an *extreme point* of S if $S \setminus i \in \mathcal{F}$. The set of extreme points of S is denoted by $ex(S)$ and $S^- = S \setminus ex(S)$. If $\mathcal{F}$ is a convex geometry and $S \in \mathcal{F}$, then the interval $[S^-, S] = \{T \in \mathcal{F} : S^- \subseteq T \subseteq S\}$ is a Boolean algebra in the lattice $(\mathcal{F}, \cup, \cap)$ (for a survey see Edelman and Jamison [5]).

If $\mathcal{F}$ is a convex geometry such that $\{i\} \in \mathcal{F}$, for all $i \in N$, and it is also a $\cup$-stable system, then $\mathcal{F}$ is called a *partition convex geometry*. The interest in the study of convex geometries comes from the search for combinatorial structures that generalize the obtained results with other structures of cooperation used in theory of games. As we have indicated in the preceding section, the dividends of the restricted game are determinant for computing the Myerson value. For any set system $\mathcal{F}$, we can obtain the dividends with the following formula

$$d_{v^{\mathcal{F}}}(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v^{\mathcal{F}}(T).$$

However, we are interested in studying conditions in which the Myerson value can be calculated in terms of the dividends in the game (N, v), and if it is possible, in terms of its characteristic function. The next theorem (see Bilbao [3]) generalizes the result obtained by Owen [14] who studies the computation of the dividends of the restricted game by a communication situation (N, v, G) where G is a tree.

Theorem 10 *Let $(N, v, \mathcal{F})$, where $\mathcal{F} \subseteq 2^N$ is a partition convex geometry and (N, v) is a game. If $S \in \mathcal{F}$, then*

$$d_{v^{\mathcal{F}}}(S) = \sum_{\{T : S \setminus ex(S) \subseteq T \subseteq S\}} (-1)^{|S|-|T|} v(T).$$

Proof. The collection $\{u_T : T \in \mathcal{F},\ T \neq \emptyset\}$ is a basis of the space of the restricted games by $\mathcal{F}$. Thus, for every $S \in \mathcal{F}$,

$$v(S) = v^{\mathcal{F}}(S) = \sum_{\{T\in\mathcal{F}:T\subseteq S\}} d_{v^{\mathcal{F}}}(T).$$

If we consider the functions $v, d_{v^{\mathcal{F}}} : \mathcal{F} \to \mathbb{R}$, and we apply the Möbius inversion function [15] to the partially ordered set $(\mathcal{F}, \subseteq)$, the dividends of $v^{\mathcal{F}}$ can be expressed in terms of the values of the game v. Indeed, for all $S \in \mathcal{F}$, we have

$$v(S) = \sum_{\{T\in\mathcal{F}:T\subseteq S\}} d_{v^{\mathcal{F}}}(T) \Longleftrightarrow d_{v^{\mathcal{F}}}(S) = \sum_{\{T\in\mathcal{F}:T\subseteq S\}} v(T)\mu(T,S).$$

The Möbius function in the case of a convex geometry [5, Theorem 4.3] is given by

$$\mu(T,S) = \begin{cases} (-1)^{|S|-|T|}, & \text{if } S\setminus T \subseteq ex(S), \\ 0, & \text{otherwise,} \end{cases}$$

and for $T \subseteq S$, we have that $S \setminus T \subseteq ex(S)$ if and only if $S \setminus ex(S) \subseteq T$. Therefore,

$$d_{v^{\mathcal{F}}}(S) = \sum_{\{T:S\setminus ex(S)\subseteq T\subseteq S\}} (-1)^{|S|-|T|} v(T). \qquad \square$$

Note that if $\mathcal{F} = 2^N$, then all elements of every coalition are extreme points. Thus, for every $S \in 2^N$, we have that $S \setminus ex(S) = \emptyset$, and as $v^{\mathcal{F}} = v$, for any $S \in 2^N$, we obtain

$$d_v(S) = d_{v^{\mathcal{F}}}(S) = \sum_{T\subseteq S} (-1)^{|S|-|T|} v(T),$$

and it is the expression of the dividends. Thus, the Myerson value of a restricted game by a partition convex geometry is given by

$$\mu_i(N,v,\mathcal{F}) = \sum_{\{S\in\mathcal{F}:i\in S\}} \frac{1}{|S|} \left[\sum_{\{T:S\setminus ex(S)\subseteq T\subseteq S\}} (-1)^{|S|-|T|} v(T) \right],$$

for all $i \in N$. The last result establishes a new algorithm which we call *dividend** that will allow us to compute the dividends of the restricted game. The description of the algorithm is as follows.

Algorithm *dividend**

$d_{v^{\mathcal{F}}}(\emptyset) \leftarrow 0$

For i **from** 1 **to** n

  For j **from** 1 **to** $S(i)$

  $d_{v^{\mathcal{F}}}\left(S_i^j\right) \leftarrow \sum_{\left\{T:S_i^j\setminus ex\left(S_i^j\right)\subseteq T\subseteq S_i^j\right\}} (-1)^{\left|S_i^j\right|-|T|} v(T)$

  end for

end for

where S_i^j is the j-th coalition such that $\left|S_i^j\right| = i$ and $S(i)$ is the number of feasible coalitions of cardinal i.

We make use of the *dividend** algorithm for computing the Myerson value in restricted games by partition convex geometries. The dividends are calculated by means of an ascending procedure which requires a previous storage of the extreme points of each feasible coalition in a table.

Theorem 11 *Let $(N, v, \mathcal{F})$ be a union stable structure, where $(N, \mathcal{F})$ is a partition convex geometry and (N, v) is a game. To compute the dividends in the restricted game, with the dividend* algorithm, requires a space $\Omega(|\mathcal{F}|)$ and a time $O\left(2^D |\mathcal{F}|\right)$, where $D = \max\{|ex(S)| : S \in \mathcal{F}\}$.*

Proof. From the description of the *dividend** algorithm, we deduce that the execution time is

$$\begin{aligned} t(dividend^*) &= 1 + t(loop1) = 1 + \sum_{i=1}^{n} t(loop2) \\ &= 1 + \sum_{i=1}^{n} \sum_{j=1}^{S(i)} t(assignment) = 1 + \sum_{i=1}^{n} \sum_{j=1}^{S(i)} t(sum) \\ &= 1 + \sum_{i=1}^{n} \sum_{j=1}^{S(i)} 2^{\left|ex(S_i^j)\right|} 2 \leq 1 + \sum_{i=1}^{n} \sum_{j=1}^{S(i)} 2^D 2 \\ &= 1 + 2^{D+1} \sum_{i=1}^{n} \sum_{j=1}^{S(i)} 1 < 1 + 2^{D+1} |\mathcal{F}|. \end{aligned}$$

Therefore, $t(dividend^*) = O\left(2^D |\mathcal{F}|\right)$. Furthermore, if we suppose that the dividends are calculated by an ascending procedure, then we obtain that the space complexity is $\Omega(|\mathcal{F}|)$. □

Theorem 12 *Let $(N, v, \mathcal{F})$ be a union stable structure, where $(N, \mathcal{F})$ is a partition convex geometry and (N, v) is a game. To compute Myerson value by dividend* requires a space $\Omega(|\mathcal{F}|)$ and a time $O\left(\max\left\{n|\mathcal{F}|, 2^D |\mathcal{F}|\right\}\right)$, where $D = \max\{|ex(S)| : S \in \mathcal{F}\}$.*

Proof. First of all, we calculate all dividends of the feasible coalitions with the *dividend** algorithm. It requires a time $O\left(2^D |\mathcal{F}|\right)$. For computing the Myerson value for the player i,

$$\mu_i(N, v, G) = \sum_{\{S \in \mathcal{F} : i \in S\}} \frac{d_{v^{\mathcal{F}}}(S)}{|S|},$$

we have that each player belongs to F_i coalitions where $F_i < |\mathcal{F}|$. The required time to evaluate the sum for every player is $O(|\mathcal{F}|)$. Thus, to compute the Myerson value for n players requires a time $O\left(\max\left\{n|\mathcal{F}|, 2^D |\mathcal{F}|\right\}\right)$. □

Finally, we analyze the complexity in the family of all connected coalitions of the path $1 - n$, which is a partition convex geometry. The purpose of studying, again, this family of feasible coalitions is to make clear the importance of using the formula of the above theorem, to obtain a significant reduction in the computation time of the dividends and the computation time of the Myerson value. Notice that the number of extreme players of each feasible coalition S is given by

$$|ex(S)| = \begin{cases} 2, & \text{if } |S| \geq 2, \\ 1, & \text{if } |S| = 1. \end{cases}$$

Theorem 13 *Let (N, v, G) be a communication situation, where G is a path $1-n$. To compute all dividends in the restricted game requires a space $\Omega\left(n^2\right)$ and a time $\Theta(n^2)$.*

Proof. For computing the dividends, we use the *dividend** algorithm and only consider the feasible coalitions. The number of coalitions with i players is given by $S(i) = n - i + 1$. Thus, the execution time of the algorithm is given by

$$\begin{aligned} t(dividend^*) &= 1 + t(loop1) = 1 + \sum_{i=1}^{n} t(loop2) \\ &= 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)} t(assignment) = 1 + \sum_{i=1}^{n}\sum_{j=1}^{S(i)} t(sum) \\ &= 1 + \sum_{j=1}^{S(1)} t(sum) + \sum_{i=2}^{n}\sum_{j=1}^{S(i)} t(sum) \\ &= 1 + \sum_{j=1}^{S(1)} 2^1\, 2 + \sum_{i=2}^{n}\sum_{j=1}^{S(i)} 2^2\, 2 \\ &= 1 + 4n^2. \end{aligned}$$

Therefore, $t(dividend^*) = \Theta(n^2)$. Since $|\mathcal{F}| = \left(n^2 + n\right)/2$, Theorem 11 implies that the required space is $\Omega\left(n^2\right)$. □

Theorem 14 *Let (N, v, G) be a communication situation, where G is a path $1-n$. To compute the Myerson value using dividend* requires a space $\Omega\left(n^2\right)$ and a time $O(n^3)$.*

Proof. If G is the path $1-n$, then the family $\mathcal{F}$ of feasible coalitions satisfies that $|\mathcal{F}| = \left(n^2 + n\right)/2$ and $D = \max\left\{|ex(S)| : S \in \mathcal{F}\right\} = 2$. Then Theorem 12 implies that the computation of the Myerson value for n players requires a time $\mathrm{O}\left(\max\left\{n\,|\mathcal{F}|\,, 2^D\,|\mathcal{F}|\right\}\right) = \mathrm{O}\left(n^3\right)$ and a space $\Omega\left(n^2\right)$. □

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