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DUAL GAMES ON COMBINATORIAL STRUCTURES

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Abstract

In the classical model of cooperative games it is generally assumed that there are no restrictions on cooperation and hence, every subset of players is a feasible coalition. However, in many social and economic situations, this model does not apply. Examples are provided by local public goods which are supplied by local communities, social and sports clubs, labor unions, political parties, and other institutions. We will define the feasible coalitions by using the dual combinatorial structures called *convex geometries* and *antimatroids*. We introduce the Shapley and Banzhaf values for these games and observe the axioms which characterize such values in detail.

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1 Introduction

Let N be a finite set of players. A pair (N, v), where $v : 2^N \to \mathbb{R}$ satisfies $v(\emptyset) = 0$, is a *game in coalitional form.* The subsets $S \in 2^N$ are called *coalitions.* The coalitions of the game are the subsets of the set N and $\left(2^N, \cup, \cap\right)$ form a Boolean algebra. An important generalization of a Boolean algebra is a distributive lattice which has the join ($\vee$) and meet ($\wedge$) operations with the same properties as in the case of Boolean algebras but without the complementation operation. It is no true that every member of the lattice is a join of single elements, but it is true that every member is a join of join-irreducible elements.

A *join-irreducible* is an element of the lattice which cannot be represented as a join of elements distinct from itself. Birkhoff [5] proved that every element of a finite distributive lattice has a unique *irredundant decomposition* as a join of join-irreducible elements. *Antimatroids* seem to have been considered first by Dilworth [8] who investigated representations of an element a in a finite and semimodular lattice as a meet $a = \bigwedge M$ of a set M of meet-irreducible elements.

This paper is organized as follows. Convex geometries and antimatroids, such as their properties, are treated in section 2. We interpret their properties in the framework of partial cooperation and explain the duality relation between them. In section 3, this duality is translated to vector spaces of the corresponding games. The Shapley and Banzhaf values are studied in sections 4 and 5. These values have been investigated by Bilbao, Jiménez-Losada and López [2] and Bilbao and Edelman [3] for games on convex geometries.

2 Convex geometries and antimatroids

We refer the reader to Korte, Lovász and Schrader [14] for a detailed treatment of combinatorial structures. *Convex geometries* are a combinatorial abstraction of convex sets introduced by Edelman and Jamison [10].

Definition 1 *A family $\mathcal{L}$ of subsets of N is a convex geometry if it satisfies the following properties:*

(C1) $\emptyset, N \in \mathcal{L}$.

(C2) If $S, T \in \mathcal{L}$ *then* $S \cap T \in \mathcal{L}$.

(C3) If $S \in \mathcal{L}$ *and* $S \neq N$, *then there exists* $j \in N \setminus S$ *such that* $S \cup j \in \mathcal{L}$.

Property (C2) implies that intersections of feasible coalitions should also be feasible, since the players agree on a profile of cooperation and (C3) is the augmentation property. We call the sets in a convex geometry *convex sets*. A *maximal chain* of $\mathcal{L} \subseteq 2^N$ is an ordered collection of convex sets that is not contained in any larger chain. Edelman and Jamison [10] showed that every maximal chain contains $n+1$ convex sets $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{n-1} \subset S_n = N$, and the cardinal $|S_i| = i$, for all $i = 0, 1, \ldots, n$. Moreover, the *hierarchical situations* by Moulin [15] when users pay their incremental costs according to an ordering of N, can be modeled by convex geometries.

The map $- : 2^N \to \mathcal{L}$ defined by

$$\overline{A} = \bigcap_{\{S \in \mathcal{L} : S \supseteq A\}} S$$

for all $A \subseteq N$ is a closure operator which satisfies $A \subseteq \overline{A}$, $\overline{\overline{A}} = \overline{A}$, and $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$, for all $A, B \subseteq N$, with the additional condition that $\overline{\emptyset} = \emptyset$.

An element i of a convex set $C \in \mathcal{L}$ is an *extreme point* of C if $C \setminus i \in \mathcal{L}$. The set of extreme points of C is denoted by $\mathrm{ex}(C)$. The convex geometries are the closure spaces satisfying the finite Minkowski-Krein-Milman property: *Every convex set is the closure of its extreme points.* It follows that $\mathrm{ex}(C) \neq \emptyset$ for every nonempty $C \in \mathcal{L}$. Furthermore, every convex geometry satisfies the following anti-exchange property:

$$\text{For all } A \subseteq N, \quad i, j \notin \overline{A}, \quad j \in \overline{A \cup i} \quad \text{imply} \quad i \notin \overline{A \cup j}.$$

This property is a combinatorial abstraction of the convex closure defined in Euclidean spaces.

Example A communication situation is a triple (N, v, G), where (N, v) is a game and $G = (N, E)$ is a graph. This concept was first introduced in Myerson [16], and investigated in Owen [17] and Borm, van den Nouweland and Tijs [7]. A graph G is a *block graph* if every block is a complete graph. If G is a forest or a tree, then G is a block graph. Jamison (see [10]) has shown that: *$G = (N, E)$ is a connected block graph if and only if the collection of subsets of N which induces connected subgraphs is a convex geometry.*

Example A subset S of a partially ordered set (poset) $(P, \leq)$ is *convex* if $a \in S$, $b \in S$ and $a \leq b$ imply that the interval $\{x \in P : a \leq x \leq b\} \subseteq S$. The convex subsets of any poset P form a convex geometry $Co(P)$ (see Birkhoff and Bennett [6]). Edelman [11] studied voting games on $Co(P)$ where $(P, \leq)$ is defined by the left-right policy order.

In a convex geometry, we can eliminate one extreme point of N, and we obtain a sequence of convex sets shelling extreme points in each step until the empty set. Normally there are several forms to make this, but all of them permit that players calculate what is happened if they leave the grand coalition. This *shelling process* relates antimatroids with convex geometries. We consider that the eliminated players in one shelling process are joined forming a sequence of coalitions. For instance, if the shelling order is $i_1, i_2, \ldots, i_n$, then they can organize the following coalitions: $\{i_1\}, \{i_1, i_2\}, \ldots, \{i_1, i_2, \ldots, i_n\}$. The set system formed in every shelling process is an antimatroid.

Definition 2 *A family $\mathcal{A}$ of subsets of N is an antimatroid if it satisfies the following properties:*

(A1) $\emptyset, N \in \mathcal{A}$.

(A2) If $S, T \in \mathcal{A}$ then $S \cup T \in \mathcal{A}$.

(A3) For $S, T \in \mathcal{A}$ with $|T| = |S| + 1$, there is an element $j \in T \setminus S$ such that $S \cup j \in \mathcal{A}$.

The map $\text{int} : 2^N \to \mathcal{A}$ defined by

$$\text{int}(A) = \bigcup_{\{S \in \mathcal{A} : S \subseteq A\}} S$$

for all $A \subseteq N$ is an interior operator such that $\text{int}(A) \subseteq A$, $\text{int}(\text{int}(A)) = \text{int}(A)$, and $A \subseteq B$ implies $\text{int}(A) \subseteq \text{int}(B)$, for all $A, B \subseteq N$, with the additional condition that $\text{int}(N) = N$. Let $S \in \mathcal{A}$. We say that $i \in N \setminus S$ is an *augmentative point* of S if $S \cup i \in \mathcal{A}$. The set of augmentative points of S is denoted by $\text{au}(S)$.

Example *Permission structures* were defined by Gilles, Owen and van den Brink [13]. They assume that players who participate in a cooperative game restricted by a hierarchical organization in which there are players that need permission from certain other players before they are allowed to cooperate. Algaba, Bilbao, van den Brink and Jiménez-Losada [1] showed that if the family $\mathcal{A}$ of subsets of N is derived from an disjunctive or conjunctive approach of an acyclic permission structure then $\mathcal{A}$ is an antimatroid.

Example Let $G = (V, E)$ be a connected graph and let $r \in V$ be one of its vertices. The collection

$$\mathcal{A} = \{S \subseteq E : (V(S), S) \text{ is a connected subgraph and } r \in V(S)\}$$

is an antimatroid called line-searched antimatroid. This antimatroid can be used to model communication networks.

Example Let $G = (V, E)$ be an undirected graph and let $r \notin V$ be a root vertex. On the ground set $V \cup \{r\}$ we define an antimatroid $(V \cup \{r\}, \mathcal{A})$ where

$$\mathcal{A} = 2^V \cup \{S \cup \{r\} : S \subseteq V \text{ covers all edges in } G\}.$$

This antimatroid allows to study what happen when a new player arrives at a game.

The shelling process lead to a relationship of duality between convex geometries and antimatroids. For an antimatroid $(N, \mathcal{A})$ we define the system of its complements $\mathcal{L} = \{C \subseteq N : N \setminus C \in \mathcal{A}\}$.

Theorem 3 *A family $\mathcal{A}$ of subsets of N is an antimatroid if and only if the system of its complements $\mathcal{L}$ is a convex geometry.*

Proof. Since $\emptyset, N \in \mathcal{A}$, we have that $\emptyset, N \in \mathcal{L}$ and conversely. $\mathcal{A}$ is closed under union if and only if $\mathcal{L}$ is closed under intersection and property (A2) for $\mathcal{A}$ is equivalent to property (C2) for $\mathcal{L}$. We only have to prove that if $\mathcal{L}$ is a convex geometry then the system of complements $\mathcal{A}$ satisfies (A3).

Let $S, T \in \mathcal{A}$ with $|S| < |T|$. Note that $S \cup j \in \mathcal{A}$ for some $j \in T \setminus S$, if and only if $T \cap \text{ex}(N \setminus S) \neq \emptyset$. If we assume that this intersection is empty, then $\text{ex}(N \setminus S) \subseteq N \setminus T$. Since both $N \setminus S$ and $N \setminus T$ are convex, the Minkowski-Krein-Milman property implies

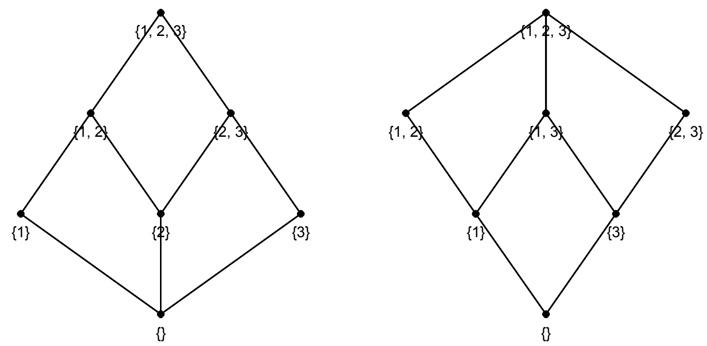
$$N \setminus S = \overline{\text{ex}(N \setminus S)} \subseteq \overline{N \setminus T} = N \setminus T,$$

contradicting that $|S| < |T|$. □

Example Let $P = (1 < 2 < 3)$ be a linear order and let us consider the convex geometry $Co(P)$. Then the collection given by

$$A(P) = \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

is the dual antimatroid of $Co(P)$.



The convex geometry $Co(P)$ and its dual antimatroid $A(P)$

In the next example, we show the unique systems that are convex geometries and antimatroids. These structures were studied by Faigle and Kern [12].

Example Let $(P, \leq)$ be a poset. The collection

$$J(P) = \{I \subseteq P : I \text{ is an order ideal}\}$$

is an antimatroid and also a convex geometry. Faigle and Kern [12] consider a partial order among the players and study games on its order ideals. A generalization of these games can be defined by the family of subsets of P, $S \subseteq P$, such that for every $i \in S$ and every chain $i_1 < i_2 < \cdots < i_k < i$ we have $S \cap \{i_1, i_2, \ldots, i_k\} \neq \emptyset$. This new set system is an antimatroid.

Given an antimatroid $\mathcal{A}$ and its dual convex geometry $\mathcal{L}$, it is easy to obtain the following property, using the definition of extreme point and augmentative point. For all $S \in \mathcal{A}$, a player i of $N \setminus S$ is an augmentative point of S if and only if i is an extreme point of $N \setminus S$, i.e.,

$$\text{au}(S) = \text{ex}(N \setminus S) \text{ for all } S \in \mathcal{A}. \tag{1}$$

3 Dual games

Let $\mathcal{F} \subseteq 2^N$ be a collection of feasible coalitions such that $\emptyset \in \mathcal{F}$ and for each $i \in N$ there exists $S \in \mathcal{F}$ with $i \in S$. We consider the vector space $\Gamma(\mathcal{F})$ of games $v : \mathcal{F} \to \mathbb{R}$ on the combinatorial structure $\mathcal{F}$. For every nonempty coalition $T \in \mathcal{F}$, the *unanimity game* ζ_T and the *identity game* δ_T, are defined as follows

$$\zeta_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise,} \end{cases} \qquad \delta_T(S) = \begin{cases} 1, & \text{if } T = S \\ 0, & \text{otherwise,} \end{cases}$$

for all $S \in \mathcal{F}$.

Proposition 4 *The collections* $\{\delta_T : T \in \mathcal{F} \setminus \{\emptyset\}\}$ *and* $\{\zeta_T : T \in \mathcal{F} \setminus \{\emptyset\}\}$ *are two bases of the vector space* $\Gamma(\mathcal{F})$.

Proof. It is easy to check that the identity games are linear independent and

$$v = \sum_{\{T \in \mathcal{F} : T \neq \emptyset\}} v(T)\, \delta_T, \tag{2}$$

for all $v \in \Gamma(\mathcal{F})$. The unanimity games are also independent because each unanimity game ζ_T can be written as

$$\zeta_T = \sum_{\{S \in \mathcal{F} : S \supseteq T\}} \delta_S. \qquad \square \tag{3}$$

In this section we construct a *dual operator* between the sets $\Gamma(\mathcal{L})$ and $\Gamma(\mathcal{A})$, that is $F : \Gamma(\mathcal{L}) \rightleftarrows \Gamma(\mathcal{A})$, where $\mathcal{A}$ is an antimatroid and $\mathcal{L}$ is its dual convex geometry. For all $v \in \Gamma(\mathcal{L})$, we have $F(v) = v^*$, where the *dual game* $v^*$of v is defined, for each $S \in \mathcal{A}$, by

$$v^*(S) = v(N) - v(N \setminus S). \tag{4}$$

Since the families $\mathcal{L}$ and $\mathcal{A}$ have the same cardinal, we deduce from Proposition 4 that the vector spaces $\Gamma(\mathcal{L})$ and $\Gamma(\mathcal{A})$ have the same dimension. Moreover, F is an isomorphism such that

$$(v^*)^* = v, \tag{5}$$

for all $v \in \Gamma(\mathcal{L})$. Therefore, using this operator and any basis of the vector space $\Gamma(\mathcal{L})$ is possible to give a new basis of the space $\Gamma(\mathcal{A})$.

Proposition 5 *Let* $\mathcal{A}$ *be an antimatroid and let* $\mathcal{L}$ *be its dual convex geometry. For every* $S \in \mathcal{A}$ *such that* $S \neq N$, *it holds:*

1. *The dual of the unanimity game* $\zeta_{N\setminus S} \in \Gamma(\mathcal{L})$ *is the game* $\mu_S \in \Gamma(\mathcal{A})$ *defined, for each* $T \in \mathcal{A}$*, by*

$$\mu_S(T) = \begin{cases} 0, & \text{if } T \subseteq S \\ 1, & \text{otherwise.} \end{cases}$$

2. *The dual of the identity game* $\delta_{N\setminus S} \in \Gamma(\mathcal{L})$, *where* $S \neq \emptyset$, *is the game* $\rho_S \in \Gamma(\mathcal{A})$ *defined, for each* $T \in \mathcal{A}$*, by*

$$\rho_S(T) = \begin{cases} -1, & \text{if } T = S \\ 0, & \text{if } T \neq S. \end{cases}$$

The dual of the game $\delta_N \in \Gamma(\mathcal{L})$ *is* $\rho_\emptyset \in \Gamma(\mathcal{A})$ *defined by* $\rho_\emptyset(T) = 1$ *for all nonempty* $T \in \mathcal{A}$.

Proof. Let $T \in \mathcal{A}$. The dual of $\zeta_{N\setminus S}$ is determined by

$$\zeta^*_{N\setminus S}(T) = \zeta_{N\setminus S}(N) - \zeta_{N\setminus S}(N \setminus T) = 1 - \begin{cases} 1, & \text{if } N \setminus T \supseteq N \setminus S \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\zeta^*_{N\setminus S}(T) = \mu_S(T)$. For the identity game $\delta_{N\setminus S}$, where $S \neq \emptyset$, we obtain

$$\delta^*_{N\setminus S}(T) = \delta_{N\setminus S}(N) - \delta_{N\setminus S}(N \setminus T) = 0 - \begin{cases} 1, & \text{if } N \setminus T = N \setminus S \\ 0, & \text{if } N \setminus T \neq N \setminus S. \end{cases}$$

Therefore, $\delta^*_{N\setminus S}(T) = \rho_S(T)$. If $S = \emptyset$ then

$$\delta^*_N(T) = \delta_N(N) - \delta_N(N \setminus T) = 1 - \begin{cases} 1, & \text{if } N \setminus T = N \\ 0, & \text{if } N \setminus T \neq N. \end{cases}$$

and hence $\delta^*_N(T) = \rho_\emptyset(T)$. □

A game $v : \mathcal{F} \to \mathbb{R}$ is *monotone* if for all $S, T \in \mathcal{F}$ with $S \subseteq T$, it holds $v(S) \leq v(T)$. A *simple* game is a monotone game $v : \mathcal{F} \to \{0, 1\}$. If $v(S) = 1$ then S is called winning coalition. Otherwise, S is losing. Now, we introduce a new concept of convexity for games on convex geometries and antimatroids.

Definition 6 *Let* $\mathcal{L}$ *be a convex geometry. A game* $v \in \Gamma(\mathcal{L})$ *is said to be convex on* $\mathcal{L}$ *if* $v\left(\overline{S \cup T}\right) + v(S \cap T) \geq v(S) + v(T)$, *for all* $S, T \in \mathcal{L}$. *A game is called concave on a convex geometry if the reverse inequality holds.*

Definition 7 *Let* $\mathcal{A}$ *be an antimatroid. A game* $v \in \Gamma(\mathcal{A})$ *is said to be convex on* $\mathcal{A}$ *if* $v(S \cup T) + v(int(S \cap T)) \geq v(S) + v(T)$, *for all* $S, T \in \mathcal{A}$. *A game is called concave on an antimatroid if the reverse inequality holds.*

Proposition 8 *Let $\mathcal{L}$ be a convex geometry and let $\mathcal{A}$ be its dual antimatroid. Then:*

1. *$v \in \Gamma(\mathcal{L})$ is monotone if and only if $v^* \in \Gamma(\mathcal{A})$ is monotone.*
2. *$v \in \Gamma(\mathcal{L})$ is simple if and only if $v^* \in \Gamma(\mathcal{A})$ is simple.*
3. *$v \in \Gamma(\mathcal{L})$ is convex if and only if $v^* \in \Gamma(\mathcal{A})$ is concave.*

Proof. Let $v \in \Gamma(\mathcal{L})$.

1. We suppose that v is monotone. If $S, T \in \mathcal{A}$, with $S \subseteq T$, then $N \setminus T \subseteq N \setminus S$ and we obtain $v(N \setminus T) \leq v(N \setminus S)$. Therefore,

$$v^*(S) = v(N) - v(N \setminus S) \leq v(N) - v(N \setminus T) = v^*(T).$$

2. Let v be a simple game. By 1 v^* is monotone and $v^*(S) = v(N) - v(N \setminus S)$, for all $S \in \mathcal{A}$. Since v is monotone, $v^*(S) \in \{0, 1\}$.

3. We suppose that v is convex. Every $A \subseteq N$ satisfies $N \setminus \operatorname{int}(A) = \overline{N \setminus A}$. Thus, for any $S, T \in \mathcal{A}$ we have

$$\begin{aligned}
& v^*(S \cup T) + v^*(\operatorname{int}(S \cap T)) \\
& = v(N) - v(N \setminus (S \cup T)) + v(N) - v(N \setminus \operatorname{int}(S \cap T)) \\
& = 2v(N) - v((N \setminus S) \cap (N \setminus T)) - v\left(\overline{(N \setminus S) \cup (N \setminus T)}\right) \\
& \leq 2v(N) - v(N \setminus S) - v(N \setminus T) = v^*(S) + v^*(T).
\end{aligned}$$

In statements 1,2 and 3, we get the reverse implications by (5). □

4 The Shapley value

A *group value* on $\Gamma(\mathcal{F})$ is a function $\Psi : \Gamma(\mathcal{F}) \to \mathbb{R}^N$. Each coordinate of Ψ is understood as the payment of the corresponding player. Let $\mathcal{L}$ be a convex geometry and let $\Psi = (\Psi_i)_{i \in N}$ be a group value on $\Gamma(\mathcal{L})$, where $\Psi_i : \Gamma(\mathcal{L}) \to \mathbb{R}$ for all $i \in N$. We consider the following axioms:

1. Linearity axiom: If $v_1, v_2 \in \Gamma(\mathcal{L})$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, then

$$\Psi(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \Psi(v_1) + \alpha_2 \Psi(v_2).$$

Definition 9 *Let $\mathcal{L}$ be a convex geometry. For any player $i \in N$, her/his marginal contribution to $S \in \mathcal{L}$ such that $i \in ex(S)$ in the game $v \in \Gamma(\mathcal{L})$ is $v\langle S, i\rangle = v(S) - v(S \setminus i)$. If $\mathcal{A}$ is an antimatroid and $v \in \Gamma(\mathcal{A})$, the marginal contribution of i to $S \in \mathcal{A}$ such that $i \in au(S)$ is given by $v\langle S, i\rangle = v(S \cup i) - v(S)$.*

Let $\mathcal{L}$ be a convex geometry and let $\mathcal{A}$ be its dual antimatroid. If $v \in \Gamma(\mathcal{L})$ and $i \in \mathrm{ex}(S)$, then by (1) and (4):

$$\begin{aligned} v^* \langle N \setminus S, i\rangle &= v^*((N \setminus S) \cup i) - v^*(N \setminus S) \\ &= v(N) - v(S \setminus i) - v(N) + v(S) \\ &= v(S) - v(S \setminus i) = v\langle S, i\rangle . \end{aligned}$$

Then for each $S \in \mathcal{L}$ with $i \in \mathrm{ex}(S)$,

$$v\langle S, i\rangle = v^*\langle N \setminus S, i\rangle . \qquad (6)$$

A player i is *dummy* in the game $v \in \Gamma(\mathcal{L})$ if, for each $S \in \mathcal{L}$ such that $i \in \mathrm{ex}(S)$,

$$v\langle S, i\rangle = \begin{cases} v(i), & \text{if } \{i\} \in \mathcal{L} \\ 0, & \text{otherwise.} \end{cases}$$

2. Dummy axiom: If $i \in N$ is a dummy player in $v \in \Gamma(\mathcal{L})$ then

$$\Psi_i(v) = \begin{cases} v(i), & \text{if } \{i\} \in \mathcal{L} \\ 0, & \text{otherwise.} \end{cases} .$$

3. Efficiency axiom: For all $v \in \Gamma(\mathcal{L})$,

$$\sum_{i \in N} \Psi_i(v) = v(N) .$$

Let $T, S \in \mathcal{L}$ with $T \subseteq S$. The number of maximal chains for the interval $[T, S]$, in the poset generated by $\mathcal{L}$, is denoted by $c([T, S])$. In the particular case of $T = \emptyset$, this number is $c(S)$, and $c(\mathcal{L})$ represents the total number of maximal chains in $\mathcal{L}$.

4. Chain axiom: For each $S \in \mathcal{L}$ and $i, j \in \mathrm{ex}(S)$,

$$c(S \setminus i)\, \Psi_j(\delta_S) = c(S \setminus j)\, \Psi_i(\delta_S) .$$

The next theorem characterizes the Shapley value $\Phi^{\mathcal{L}}$ on convex geometries (see Bilbao [4]).

Theorem 10 *Let $\mathcal{L}$ be a convex geometry. There exists a unique group value $\Phi^{\mathcal{L}}$ on $\Gamma(\mathcal{L})$ that satisfies linearity, dummy, efficiency and chain axioms. Moreover, for all $i \in N$ and each game $v \in \Gamma(\mathcal{L})$,*

$$\Phi_i^{\mathcal{L}}(v) = \sum_{\{S \in \mathcal{L} : i \in \mathrm{ex}(S)\}} \frac{c(S \setminus i)\; c([S, N])}{c(\mathcal{L})}\, v\langle S, i\rangle .$$

We propose a new concept of dummy player on antimatroids and convex geometries.

Definition 11 *Let $\mathcal{A}$ be an antimatroid. A player $i \in N$ is a c-dummy player if for all $S, T \in \mathcal{A}$ with $i \in au(S) \cap au(T)$, we have $v\langle S, i\rangle = v\langle T, i\rangle$.*

If i is a c-dummy player then his/her marginal contributions are equal, and we denote this value by $v\langle i\rangle$. In particular, when $\{i\} \in \mathcal{A}$ and i is c-dummy, we have $v\langle i\rangle = v(i)$. So, we write the following axiom for a value Ψ on $\Gamma(\mathcal{A})$.

C-dummy axiom: If $i \in N$ is a c-dummy player in the game $v \in \Gamma(\mathcal{A})$ then

$$\Psi_i(v) = v\langle i\rangle .$$

In a convex geometry, a player is a c-dummy player if all the marginal contributions are equal. The Shapley value obtained by Bilbao [4] satisfies the c-dummy axiom for games on convex geometries.

Proposition 12 *Let $\mathcal{L}$ be a convex geometry. A group value Ψ on $\Gamma(\mathcal{L})$ satisfies linearity, c-dummy, efficiency and chain axioms if and only if $\Psi = \Phi^{\mathcal{L}}$.*

Proof. We only need to prove that the c-dummy axiom implies the dummy axiom. If i is a dummy player then

$$v\langle i\rangle = \begin{cases} v(i), & \text{if } \{i\} \in \mathcal{L} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, using the c-dummy axiom, $\Psi_i(v) = v\langle i\rangle$. Thus Ψ satisfies linearity, dummy, efficiency and chain axioms. Theorem 10 implies that this value is unique and hence $\Psi = \Phi^{\mathcal{L}}$. $\square$

Proposition 13 *Let $\mathcal{L}$ be a convex geometry and let $\mathcal{A}$ be its dual antimatroid. For any group value $\Psi^{\mathcal{L}} : \Gamma(\mathcal{L}) \to \mathbb{R}^N$ we define the group value on $\Gamma(\mathcal{A})$ by $\Psi^{\mathcal{A}} = \Psi^{\mathcal{L}} \circ F$. Then the following statements hold.*

1. *$\Psi^{\mathcal{L}}$ satisfies the linearity axiom if and only if $\Psi^{\mathcal{A}}$ satisfies the linearity axiom.*

2. *$\Psi^{\mathcal{L}}$ satisfies the dummy player axiom if and only if $\Psi^{\mathcal{A}}$ satisfies dummy player axiom.*

3. *$\Psi^{\mathcal{L}}$ satisfies the efficiency axiom if and only if $\Psi^{\mathcal{A}}$ satisfies the efficiency axiom.*

Proof. 1. Since F is an isomorphism and $\Psi^{\mathcal{L}}$ is linear then $\Psi^{\mathcal{A}}$ is linear too.

2. In fact, we claim that if i is a dummy player in $v \in \Gamma(\mathcal{A})$ then i is a dummy player in $v^* \in \Gamma(\mathcal{L})$. Let $S, T \in \mathcal{L}$ and $i \in \mathrm{ex}(S) \cap \mathrm{ex}(T)$. Thus, by (1), we obtain $i \in \mathrm{au}(N \setminus S) \cap \mathrm{au}(N \setminus T)$ where $N \setminus S, N \setminus T \in \mathcal{A}$. Using (6) we have

$$v^* \langle S, i \rangle = v \langle N \setminus S, i \rangle = v \langle N \setminus T, i \rangle = v^* \langle T, i \rangle$$

because i is dummy in v. Moreover, $v^* \langle i \rangle = v \langle i \rangle$. Finally, if i is a dummy player in v then

$$\Psi^{\mathcal{A}}(v) = \Psi^{\mathcal{L}}(v^*) = v^* \langle i \rangle = v \langle i \rangle .$$

3. For each $v \in \Gamma(\mathcal{A})$ we have that $v^*(N) = v(N) - v(\emptyset) = v(N)$. If $\Psi^{\mathcal{L}}$ is efficient then $\Psi^{\mathcal{L}}(w) = w(N)$ for all $w \in \Gamma(\mathcal{L})$. Hence, we have, for all $v \in \Gamma(\mathcal{A})$,

$$\Psi^{\mathcal{A}}(v) = \Psi^{\mathcal{L}}(v^*) = v^*(N) = v(N) .$$

Equality (5) implies the reverse implications. □

It remains to analyze as the chain axiom is translated to antimatroids by duality. For $T, S \in \mathcal{A}$ with $T \subseteq S$, we denote by $c([T, S])$ the number of maximal chains for $[T, S]$ in the poset $(\mathcal{A}, \subseteq)$, in the same way as in a convex geometry. Then $c(\mathcal{A})$ represents the number of maximal chains in $(\mathcal{A}, \subseteq)$.

Proposition 14 *Let $\mathcal{L}$ be a convex geometry and let $\mathcal{A}$ be its dual antimatroid. Then $\Psi^{\mathcal{L}}$ is a group value on $\Gamma(\mathcal{L})$ satisfying the chain axiom if and only if the value $\Psi^{\mathcal{A}} = \Psi^{\mathcal{L}} \circ F$ satisfies that, for all $S \in \mathcal{A}$, $S \neq N$, and $i, j \in au(S)$,*

$$c([S \cup i, N]) \; \Psi_j^{\mathcal{A}}(\rho_S) = c([S \cup j, N]) \; \Psi_i^{\mathcal{A}}(\rho_S) .$$

Proof. Let $S \in \mathcal{A}$, $S \neq N$. By Proposition 5 $\rho_S = \delta^*_{N \setminus S}$ where $N \setminus S \in \mathcal{L}$ and $N \setminus S \neq \emptyset$. Therefore, (5) implies $\Psi_i^{\mathcal{A}}(\rho_S) = \Psi_i^{\mathcal{L}}\left(\delta_{N \setminus S}\right)$. If $i, j \in \mathrm{au}(S)$ then $i, j \in \mathrm{ex}(N \setminus S)$ by (1) and we can substitute these numbers in the chain axiom.

On the other hand, if $S \in \mathcal{L}$ then it is easy to see that there exists an isomorphism between the maximal chains of the interval $[\emptyset, S]$ in the convex geometry and the maximal chains of the interval $[N \setminus S, N]$ in its dual antimatroid. If S is formed in $\mathcal{L}$ then the rest of players form the coalition $N \setminus S$ in $\mathcal{A}$. Thus, depending on the individual eliminations given by the chain, the players will join to $N \setminus S$. Therefore,

$$c(S \setminus i) = c([S \cup i, N]) ,$$
$$c(S \setminus j) = c([S \cup j, N]) .$$

The other implication is due to (5). □

The above proposition allows us to define a new axiom for games on antimatroids.

$\mathcal{A}$-chain axiom: For each $S \in \mathcal{A}$, $S \neq N$, and for any $i, j \in \mathrm{au}(S)$,

$$c([S \cup i, N])\ \Psi_j(\rho_S) = c([S \cup j, N])\ \Psi_i(\rho_S).$$

The last axiom can be interpreted as follows: the players that obtain positive marginal contributions in the game ρ_S are those that are join to S immediately, and their payments are in proportion to the number of feasible orders. The above axioms characterizes the Shapley value on antimatroids.

Theorem 15 *Let $\mathcal{A}$ be an antimatroid. There exists a unique group value $\Phi^{\mathcal{A}}$ on $\Gamma(\mathcal{A})$ that satisfies linearity, dummy, efficiency and $\mathcal{A}$-chain axioms. Moreover, for all $i \in N$, and each game $v \in \Gamma(\mathcal{A})$,*

$$\Phi_i^{\mathcal{A}}(v) = \sum_{\{S \in \mathcal{A} : i \in \mathrm{au}(S)\}} \frac{c(S)\ c([S \cup i, N])}{c(\mathcal{A})} v\langle S, i\rangle.$$

This value is called the Shapley value of $v \in \Gamma(\mathcal{A})$.

Proof. We first prove the existence and uniqueness of $\Phi^{\mathcal{A}}$. The value $\Phi^{\mathcal{A}} = \Phi^{\mathcal{L}} \circ F$, where $\Phi^{\mathcal{L}}$ is the Shapley value on $\Gamma(\mathcal{L})$, is the only value that satisfies linearity, dummy, efficiency and $\mathcal{A}$-chain axioms by Propositions 13 and 14.

We will get now the formula that represents the value $\Phi^{\mathcal{A}}$. If $v \in \Gamma(\mathcal{A})$ then

$$\Phi_i^{\mathcal{A}}(v) = \left(\Phi^{\mathcal{L}} \circ F\right)_i(v) = \Phi_i^{\mathcal{L}}(v^*).$$

and by Theorem 10,

$$\begin{aligned}\Phi_i^{\mathcal{A}}(v) &= \sum_{\{T \in \mathcal{L} : i \in \mathrm{ex}(T)\}} \frac{c(T \setminus i)\ c([T, N])}{c(\mathcal{L})} v^*\langle T, i\rangle \\ &= \sum_{\{T \in \mathcal{L} : i \in \mathrm{ex}(T)\}} \frac{c(T \setminus i)\ c([T, N])}{c(\mathcal{L})} v\langle N \setminus T, i\rangle.\end{aligned}$$

Note that (1) implies that there exists an isomorphism between the collections $\{T \in \mathcal{L} : i \in \mathrm{ex}(T)\}$ and $\{S \in \mathcal{A} : i \in \mathrm{au}(S)\}$, identifying T with $S = N \setminus T$. Hence,

$$\Phi_i^{\mathcal{A}}(v) = \sum_{\{S \in \mathcal{A} : i \in \mathrm{au}(S)\}} \frac{c(S)\ c([S \cup i, N])}{c(\mathcal{A})} v\langle S, i\rangle,$$

because we check the isomorphism between the maximal chains in $[\emptyset, T]$ and the maximal chains in $[N \setminus T, N]$. $\square$

5 The Banzhaf index

Let $\mathcal{L}$ be a convex geometry and let $\Gamma_s(\mathcal{L})$ be the set of the simple games on $\mathcal{L}$. Then $\Gamma_s(\mathcal{L})$ is a distributive lattice with the operations:

$$(v \wedge w)(S) = \min\{v(S), w(S)\}, \quad (v \vee w)(S) = \max\{v(S), w(S)\},$$

for all $v, w \in \Gamma_s(\mathcal{L})$ and $S \in \mathcal{L}$.

The Banzhaf index on convex geometries was studied in Bilbao et al. [2]. Given a simple game on $\mathcal{L}$, the Banzhaf index for a player i can be defined as the number of winning coalitions in which is necessary her/his presence to win.

Definition 16 *A convex swing of a player i in $v \in \Gamma_s(\mathcal{L})$ is a pair $(S, S \setminus i)$ such that $S \in \mathcal{L}$, with $i \in ex(S)$, $v(S) = 1$, and $v(S \setminus i) = 0$.*

The number of convex swings of player i is denoted by $cs_i(v)$ and it is called *Banzhaf index* for i. The total number of convex swings in v is $cs(v) = \sum_{i \in N} cs_i(v)$.

For each player $i \in N$, we define its *extreme power* by

$$\mathcal{E}_i = |\{T \in \mathcal{L} : i \in \mathrm{ex}(T)\}|,$$

i.e., the number of convex sets in which i is extreme point. For each player $i \in N$, the *extreme power on $S \in \mathcal{L}$* is the number

$$\mathcal{E}_i(S) = |\{T \in \mathcal{L} : i \in \mathrm{ex}(T),\ S \subseteq T\}|.$$

Bilbao et al. [2] defined the following four axioms for a group value $\varphi : \Gamma_s(\mathcal{L}) \to \mathbb{R}^N$.

Transfer axiom: If $v, w \in \Gamma_s(\mathcal{L})$, then

$$\varphi(v) + \varphi(w) = \varphi(v \vee w) + \varphi(v \wedge w).$$

Null player axiom If $i \in N$ is a null player in $v \in \Gamma_s(\mathcal{L})$, that is $v\langle i \rangle = 0$, then $\varphi_i(v) = 0$.

The above axiom is named dummy player in the classical model of total cooperation by Dubey and Shapley [9].

Total swings axiom: For all $v \in \Gamma_s(\mathcal{L})$,

$$\sum_{i \in N} \varphi_i(v) = cs(v).$$

The next axiom replaces the classical symmetry axiom.

Extreme power axiom: For all $S \in \mathcal{L}$, and for any $i, j \in \mathrm{ex}(S)$,

$$\mathcal{E}_i(S)\ \varphi_j(\zeta_S) = \mathcal{E}_j(S)\ \varphi_i(\zeta_S).$$

Theorem 17 *Let $\mathcal{L}$ be a convex geometry. The Banzhaf index is the unique value on $\Gamma_s(\mathcal{L})$ that satisfies transfer, null player, total swings and extreme power axioms.*

We introduce the following concepts for simple games on antimatroids.

Definition 18 *Let $\mathcal{A}$ an antimatroid and let $v \in \Gamma_s(\mathcal{A})$. A feasible swing of i is a pair $(T, T \cup i)$ such that $T \in \mathcal{A}$, $i \in au(T)$, $v(T) = 0$, and $v(T \cup i) = 1$.*

The number of feasible swings of player i is denoted by $as_i(v)$ and this is the *Banzhaf index* for i. We define $as(v) = \sum_{i \in N} as_i(v)$. By Proposition 8 we have that if v is simple then v^* is simple.

Proposition 19 *Let $\mathcal{L}$ be a convex geometry and let $\mathcal{A}$ be its dual antimatroid. Then $as_i(v^*) = cs_i(v)$ for all $v \in \Gamma_s(\mathcal{L})$.*

Proof. We only need to check that, given $i \in N$, the pair $(S, S \cup i)$ is a feasible swing of i in v^* if and only if $(N \setminus S, (N \setminus S) \setminus i)$ is a convex swing of i in v. By definition

$$as_i(v^*) = \sum_{\{T \in \mathcal{A}: i \in \mathrm{au}(T)\}} v^* \langle T, i \rangle,$$
$$cs_i(v) = \sum_{\{S \in \mathcal{L}: i \in \mathrm{ex}(S)\}} v \langle S, i \rangle.$$

Therefore, we obtain the result by using (6) and the biyection between the sets $\{T \in \mathcal{A} : i \in \mathrm{au}(T)\}$ and $\{S \in \mathcal{L} : i \in \mathrm{ex}(S)\}$. □

We will study how the above axioms are related by duality.

Proposition 20 *Let $\mathcal{L}$ be a convex geometry and let $\mathcal{A}$ be its dual antimatroid. For any group value $\Psi^{\mathcal{L}} : \Gamma_s(\mathcal{L}) \to \mathbb{R}^N$ we define the group value on $\Gamma_s(\mathcal{A})$ by $\Psi^{\mathcal{A}} = \Psi^{\mathcal{L}} \circ F$. Then the following statements hold.*

1. *$\Psi^{\mathcal{L}}$ satisfies the transfer axiom if and only if $\Psi^{\mathcal{A}}$ satisfies the transfer axiom.*

2. *$\Psi^{\mathcal{L}}$ satisfies the null player axiom if and only if $\Psi^{\mathcal{A}}$ satisfies the null player axiom.*

3. *$\Psi^{\mathcal{L}}$ satisfies the total swings axiom if and only if $\Psi^{\mathcal{A}}$ satisfies the total swings axiom.*

Proof. 1. We consider $v, w \in \Gamma_s(\mathcal{A})$. If both of them are not null, $v, w \neq 0$ then we will prove $(v \vee w)^* = v^* \wedge w^*$ and $(v \wedge w)^* = v^* \vee w^*$. When a feasible coalition S is winning in v then the dual coalition $N \setminus S$ is losing in v^*, because $v(N) = 1$. In the following table we check the first equality (the other equality is obtained in the same way). For any $S \in \mathcal{A}$, we have

v	w	$v \vee w$	$(v \vee w)^*$	v^*	w^*	$v^* \wedge w^*$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	0	0	1	1	1	1

Then we obtain that

$$\begin{aligned}\Psi^{\mathcal{A}}(v \vee w) + \Psi^{\mathcal{A}}(v \wedge w) &= \Psi^{\mathcal{L}}((v \vee w)^*) + \Psi^{\mathcal{L}}((v \wedge w)^*) \\ &= \Psi^{\mathcal{L}}(v^* \wedge w^*) + \Psi^{\mathcal{L}}(v^* \vee w^*) \\ &= \Psi^{\mathcal{L}}(v^*) + \Psi^{\mathcal{L}}(w^*) = \Psi^{\mathcal{A}}(v) + \Psi^{\mathcal{A}}(w).\end{aligned}$$

2. This proof is similar to the equivalence obtained in Theorem 13 (2).
3. By Proposition 19 we have that $as(v) = cs(v^*)$ and hence,

$$\sum_{i \in N} \Psi_i^{\mathcal{A}}(v) = \sum_{i \in N} \Psi_i^{\mathcal{L}}(v^*) = cs(v^*) = as(v).$$

We get the other implication by (5). □

Let $\mathcal{A}$ be an antimatroid. For each player $i \in N$, we define the *augmentative power* and the *augmentative power on* $S \in \mathcal{A}$, by

$$\mathcal{X}_i = |\{R \in \mathcal{A} : i \in \mathrm{au}(R)\}|,$$
$$\mathcal{X}_i(S) = |\{R \in \mathcal{A} : i \in \mathrm{au}(R), R \subseteq S\}|.$$

Proposition 21 *Let $\mathcal{L}$ be a convex geometry and let $\mathcal{A}$ be its dual antimatroid. The value $\Psi^{\mathcal{L}}$ on $\Gamma_s(\mathcal{L})$ satisfies the extreme power axiom if and only if $\Psi^{\mathcal{A}} = \Psi^{\mathcal{L}} \circ F$ satisfies that, for any coalition $S \in \mathcal{A}$ and $i, j \in au(S)$,*

$$\mathcal{X}_i(S)\ \Psi_j^{\mathcal{A}}(\mu_S) = \mathcal{X}_j(S)\ \Psi_i^{\mathcal{A}}(\mu_S).$$

Proof. Since $\mathcal{L}$ and $\mathcal{A}$ are duals (1) implies that there exists a biyection between $\{T \in \mathcal{A} : i \in \mathrm{au}(T), T \subseteq S\}$ and $\{R \in \mathcal{L} : i \in \mathrm{ex}(R), N \setminus S \subseteq R\}$. This allows to affirm that, for all $S \in \mathcal{L}$,

$$\mathcal{X}_i(S) = \mathcal{E}_i(N \setminus S). \tag{7}$$

Since $\mu_S = F(\zeta_{N \setminus S})$ by Proposition 5 we have $\Psi_i^{\mathcal{A}}(\mu_S) = \Psi_i^{\mathcal{L}}(\zeta_{N \setminus S})$ for all $i \in N$. Thus,

$$\mathcal{E}_i(N \setminus S)\, \Psi_j^{\mathcal{L}}(\zeta_{N \setminus S}) = \mathcal{E}_j(N \setminus S)\, \Psi_i^{\mathcal{L}}(\zeta_{N \setminus S}).$$

This equality follows from the extreme power axiom applied to $\Psi^{\mathcal{L}}$ (if $i, j \in \mathrm{au}(S)$ then $i, j \in \mathrm{ex}(N \setminus S)$ by (1)). □

Let $\mathcal{A}$ be an antimatroid and $\Psi^{\mathcal{A}} : \Gamma_s(\mathcal{A}) \to \mathbb{R}^N$ a group value.

Augmentative power axiom: For any coalition $S \in \mathcal{A}$ and $i, j \in \mathrm{au}(S)$,

$$\mathcal{X}_i(S)\ \Psi_j^{\mathcal{A}}(\mu_S) = \mathcal{X}_j(S)\ \Psi_i^{\mathcal{A}}(\mu_S).$$

This axiom says that the value of the game μ_S is proportional to the augmentative power of the player in the coalition S. We can state the following result on antimatroids.

Theorem 22 *Let $\mathcal{A}$ be an antimatroid. The Banzhaf index is the unique value on $\Gamma_s(\mathcal{A})$ that satisfies transfer, null player, total swings and augmentative power axioms.*

The *normalized Banzhaf index* is defined, for each $v \in \Gamma_s(\mathcal{L})$, by

$$\beta_i^{\mathcal{L}}(v) = \frac{cs_i(v)}{cs(v)} \text{ for all } i \in N.$$

We can define this index on antimatroids by $\beta^{\mathcal{A}} = \beta^{\mathcal{L}} \circ F$ and hence, for each $v \in \Gamma_s(\mathcal{A})$,

$$\beta_i^{\mathcal{A}}(v) = \frac{as_i(v)}{as(v)} \text{ for all } i \in N.$$

We consider the relationship between the convex swings of a player and the coalitions where this player is an extreme point. Let $\mathcal{L}$ a convex geometry and let $v \in \Gamma_s(\mathcal{L})$. The *probabilistic Banzhaf index* is defined, for each $v \in \Gamma_s(\mathcal{L})$, by

$$\beta_i'^{\mathcal{L}}(v) = \frac{cs_i(v)}{\mathcal{E}_i} \text{ for all } i \in N.$$

The quotient $\mathcal{E}_i(S)/\mathcal{E}_i$ is called the *relative extreme power of i on S*. Bilbao et al. [2] proposed the following axioms for this index:

Total swing probabilities: For all $v \in \Gamma_s(\mathcal{L})$,

$$\sum_{i \in N} \Psi_i(v) = \sum_{i \in N} \beta_i'^{\mathcal{L}}(v).$$

Relative extremal power axiom: For each $S \in \mathcal{L}$, and for any $i, j \in \mathrm{ex}(S)$,

$$\frac{\mathcal{E}_i(S)}{\mathcal{E}_i} \Psi_j(\zeta_S) = \frac{\mathcal{E}_j(S)}{\mathcal{E}_j} \Psi_i(\zeta_S).$$

Theorem 23 *Let $\mathcal{L}$ be a convex geometry. The probabilistic Banzhaf index $\beta'^{\mathcal{L}}$ is the unique value on $\Gamma_s(\mathcal{L})$ that satisfies transfer, null player, total swing probabilities and relative extreme power axioms.*

Similarly, given an antimatroid $\mathcal{A}$, the swing probability of player i in the game $v \in \Gamma_s(\mathcal{A})$ is $\beta_i'^{\mathcal{A}}(v) = as_i(v)/\mathcal{X}_i$, and this is the *probabilistic Banzhaf index* for i. We call *relative augmentative power* of i to the number $\mathcal{X}_i(S)/\mathcal{X}_i$.

From (7) with $S = N$ we obtain that $\mathcal{X}_i = \mathcal{E}_i$, i.e., the augmentative and extreme powers of a player are equal in an antimatroid and its dual convex geometry. Thus, if $\Psi^{\mathcal{L}}$ is a value on $\Gamma_s(\mathcal{L})$ satisfying relative extreme power axiom then $\Psi^{\mathcal{A}} = \Psi^{\mathcal{L}} \circ F$ verifies the following axiom.

Relative augmentative power axiom: For each $S \in \mathcal{A}$, and for any $i, j \in \mathrm{au}(S)$,

$$\frac{\mathcal{X}_i(S)}{\mathcal{X}_i}\Psi_j(\mu_S) = \frac{\mathcal{X}_j(S)}{\mathcal{X}_j}\Psi_i(\mu_S).$$

Proposition 20 implies the following result.

Theorem 24 *Let $\mathcal{A}$ be an antimatroid. The probabilistic Banzhaf index $\beta'^{\mathcal{A}}$ is the unique value on $\Gamma_s(\mathcal{A})$ that verifies transfer, null player, total swing probabilities and relative augmentative power axioms.*

Remark 25 *Note that the probabilistic Banzhaf index satisfies, for all $i \in N$ and $v \in \Gamma_s(\mathcal{A})$,*

$$\beta_i'^{\mathcal{A}}(v) = \sum_{\{S \in \mathcal{A} : i \in \mathrm{au}(S)\}} \frac{1}{\mathcal{X}_i} v\langle S, i\rangle,$$

where

$$\sum_{\{S \in \mathcal{A} : i \in \mathrm{au}(S)\}} \frac{1}{\mathcal{X}_i} = 1 \text{ and } \mathcal{X}_i > 0.$$

References

[1] E. Algaba., J.M Bilbao, R. van den Brink and A. Jiménez-Losada, Cooperative games on antimatroids (2002), available at http://www.esi2.us.es/~mbilbao/pdffiles/antimat.pdf

[2] J.M. Bilbao, A. Jiménez-Losada and J.J. López, The Banzhaf power index on convex geometries, Mathematical Social Sciences, 36 (1998), 157–173.

[3] J.M. Bilbao and P.H. Edelman, The Shapley value on convex geometries, Discrete Applied Mathematics, 103 (2000), 33-40.

[4] J.M. Bilbao, Cooperative games on combinatorial structures, Kluwer Academic Publishers, Boston (2000).

[5] G. Birkhoff, Ring of sets, Duke Mathematical Journal, 3 (1937), 442–454.

[6] G. Birkhoff and M.K. Bennett, The convexity lattice of a poset, Order, 2 (1985), 223–242.

[7] P. Borm, A van den Nouweland and S.H. Tijs, Cooperation and communication restrictions: a survey. In: R.P. Gilles and P.H.M. Ruys (Eds.), Imperfections and Behavior in Economic Organizations, Kluwer Academic Publishers, Boston (1994), pp. 195–227.

[8] R.P. Dilworth, Lattices with unique irreducible decompositions, Annals of Mathematics, 21 (1940), 771–777.

[9] P. Dubey and L.S. Shapley, Mathematical properties of the Banzhaf power index, Mathematics of Operations Research, 4 (1979), 99–131.

[10] P.H. Edelman and R.E. Jamison, The theory of convex geometries, Geometriae Dedicata, 19 (1985), 247–270.

[11] P.H. Edelman, A note on voting, Mathematical Social Sciences, 34 (1997), 37–50.

[12] U. Faigle and W. Kern, The Shapley value for cooperative games under precedence constraints, International Journal of Game Theory, 21 (1992), 249–266.

[13] R.P. Gilles, G. Owen and R. van den Brink, Games with permission structures: the conjunctive approach, International Journal of Game Theory, 20 (1992), 277–293.

[14] B. Korte, L. Lovász and R. Schrader, Greedoids, Springer-Verlag, Berlin (1991).

[15] H. Moulin, Incremental cost sharing: characterization by strategy-proofness, Social Choice and Welfare, 16 (1999), 279–320.

[16] R.B. Myerson, Graphs and cooperation in games, Mathematics of Operations Research, 2 (1977), 225–229.

[17] G. Owen, Values of graph-restricted games, SIAM Journal in Algebraic and Discrete Methods, 7 (1986), 210–220.

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