ELSEVIER

European Journal of Operational Research 143 (2002) 181–196

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

www.elsevier.com/locate/dsw

Decision Aiding

Voting power in the European Union enlargement

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Received 30 October 2000; accepted 14 September 2001

Abstract

The Shapley–Shubik power index in a voting situation depends on the number of orderings in which each player is pivotal. The Banzhaf power index depends on the number of ways in which each voter can effect a swing. If there are n players in a voting situation, then the function which measures the worst case running time for computing these indices is in $O(n2^n)$. We present a combinatorial method based in *generating functions* to compute these power indices efficiently in weighted double or triple majority games and we study the *time complexity* of the algorithms. Moreover, we calculate these power indices for the countries in the Council of Ministers of the European Union under the new decision rules prescribed by the Treaty of Nice.

Keywords: Power indices; Generating function; Computational complexity

1. Introduction

A simple game is a conflict in which the only objective is winning and the only rule is an algorithm to decide which coalitions are winning. These games have been used to study the distribution of power in voting situations. Two power indices have received the most theoretical attention as well as application to political structures. The first such power index was proposed by Shapley and Shubik [8] who apply the Shapley value [7] to the case of simple games. The second power index was introduced by Banzhaf [1] and has been used in arguments in various legal proceedings.

The computation of these power indices is complex in practice because the algorithms have exponential complexity. However, using generating functions, Cantor (1962) (see [5]) and Brams and Affuso [3] have obtained significative results for computing the Shapley–Shubik and the Banzhaf indices in weighted voting games. Starting from these ideas we define algorithms to compute these indices in weighted double or triple majority games.

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PII: S0377-2217(01)00334-4

In Section 2 we introduce the concept of generating function to solve the problem of counting the coalitions that have certain properties. Sections 3 and 4 are devoted to describe the computation of the normalized Banzhaf and the Shapley–Shubik indices of weighted m-majority games by using generating functions. The starting point of this approach are in works due to Cantor, Brams and Affuso. Taking into account that the two decision rules approved in the Treaty of Nice are triple majority voting games, in Section 5 we calculate the Banzhaf and Shapley–Shubik indices for the European Union enlargement under the Nice rules.

2. Generating functions

In order to obtain the normalized Banzhaf and the Shapley–Shubik power indices exactly, we present a combinatorial method based in the generating functions. The most useful method for counting the number $f(n)$ of elements of finite sets S_n, where $n \in \mathbb{N}$, is to obtain its *generating function* $F(x) = \sum_{n \geqslant 0} f(n)x^n$.

For each $n \in \mathbb{N}$, the number of subsets of k elements of $N = \{1, \dots, n\}$ is given by the explicit formula of the binomial coefficients

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

A generating function approach to binomial coefficients may be obtained as follows. Let $S = \{x_1, x_2, \dots, x_n\}$ be an n-element set. Let us regard the elements $x_1, x_2, \dots, x_n$ as independent indeterminates. It is a consequence of the process of multiplication that

$$(1+x_1)(1+x_2)\cdots(1+x_n) = \sum_{T \subseteq S} \prod_{x_i \in T} x_i.$$

Note that if $T = \emptyset$ then we obtain 1. If we put each $x_i = x$, we obtain

$$(1+x)^n = \sum_{T \subseteq S} \prod_{x \in T} x = \sum_{T \subseteq S} x^{|T|} = \sum_{k \geqslant 0} \binom{n}{k} x^k.$$

Example 1. The Fibonacci numbers are $a_0 = a_1 = 1,\ a_n = a_{n-1} + a_{n-2}$ if $n \geqslant 2$. Then

$$\begin{aligned} F(x) &= \sum_{n \geqslant 0} a_n x^n = 1 + x + \sum_{n \geqslant 2} a_n x^n = 1 + x + \sum_{n \geqslant 2} (a_{n-1} + a_{n-2}) x^n \\ &= 1 + x + x \sum_{n \geqslant 2} a_{n-1} x^{n-1} + x^2 \sum_{n \geqslant 2} a_{n-2} x^{n-2} = 1 + x + x(F(x) - 1) + x^2 F(x) = 1 + (x + x^2) F(x). \end{aligned}$$

Therefore, its generating function is

$$F(x) = \frac{1}{1 - x - x^2}.$$

We will work with generating functions of several variables

$$F(x,y,z) = \sum_{k \geqslant 0} \sum_{j \geqslant 0} \sum_{l \geqslant 0} f(k,j,l) x^k y^j z^l.$$

3. The Banzhaf index

A *simple game* is a pair (N, v), where $N = \{1, \dots, n\}$ is the set of players and $v : 2^N \to \{0, 1\}$ is the characteristic function which satisfies $v(S) \leqslant v(T)$ whenever $S \subseteq T$. A coalition is *winning* if $v(S) = 1$, and *losing* if $v(S) = 0$. The collection of all winning coalitions is denoted by $\mathscr{W}$. We will use a shorthand notation and write $S \cup i$ for the set $S \cup \{i\}$, and $S \setminus i$ for $S \setminus \{i\}$.

We introduce a special class of simple games called *weighted voting games*. The symbol $[q; w_1, \ldots, w_n]$ will be used, where q and $w_1, \ldots, w_n$ are positive integers with $w_i < q \leqslant \sum_{i=1}^n w_i$, for $i = 1, \ldots, n$. Here there are n players, w_i is the number of votes of player i, and q is the quota needed for a coalition to win. Then, the above symbol represents the simple game (N, v) defined by

$$v(S) = \begin{cases} 1 & \text{if } w(S) \geqslant q, \\ 0 & \text{if } w(S) < q, \end{cases}$$

where $S \subseteq N$ and $w(S) = \sum_{i \in S} w_i$. Given the simple games $(N, v_1), \ldots, (N, v_m)$ we consider the simple game $(N, v_1 \wedge \cdots \wedge v_m)$ defined by

$$(v_1 \wedge \cdots \wedge v_m)(S) = \min\{v_t(S) : 1 \leqslant t \leqslant m\}.$$

A *weighted m-majority game* is the simple game $(N, v_1 \wedge \cdots \wedge v_m)$, where the games (N, v_t) are the weighted voting games represented by $[q^t; w_1^t, \ldots, w_n^t]$ for $1 \leqslant t \leqslant m$. Then, its characteristic function is given by

$$(v_1 \wedge \cdots \wedge v_m)(S) = \begin{cases} 1 & \text{if } w^t(S) \geqslant q^t, \quad 1 \leqslant t \leqslant m, \\ 0 & \text{otherwise,} \end{cases}$$

where $w^t(S) = \sum_{i \in S} w_i^t$. If $m = 2$ or $m = 3$ then we obtain weighted double or triple majority game, respectively.

Let us consider a simple game (N, v). The Banzhaf index concerns the number of times each player could change a coalition from losing to winning and it requires to know the number of swings for each player i. A *swing* for player i is a pair of coalitions $(S \cup i, S)$ such that $S \cup i$ is winning and S is not, that is, the number of winning coalitions in which player i is pivotal. For each $i \in N$, we denote by $\eta_i(v)$ the number of swings for i in game v, and the total number of swings is $\overline{\eta}(v) = \sum_{i \in N} \eta_i(v)$.

The normalized *Banzhaf index* is the vector $\beta(v) = (\beta_1(v), \ldots, \beta_n(v))$, given by

$$\beta_i(v) = \frac{\eta_i(v)}{\overline{\eta}(v)}, \quad 1 \leqslant i \leqslant n.$$

We now present generating functions for computing the Banzhaf power index in weighted double majority games.

Proposition 1. *Let (N, v) be a weighted double majority game with $v = v_1 \wedge v_2$, where $v_1 = [q; w_1, \ldots, w_n]$ and $v_2 = [l; p_1, \ldots, p_n]$. For every $i \in N$, the number of swings for player i is given by*

$$\eta_i(v) = \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=l-p_i}^{p(N\setminus i)} b_{kr}^i - \sum_{k=q}^{w(N\setminus i)} \sum_{r=l}^{p(N\setminus i)} b_{kr}^i,$$

where b_{kr}^i is the number of coalitions S such that $i \notin S$ with $w(S) = k$ and $p(S) = r$.

Proof. First of all, we consider the set of all coalitions S such that $i \notin S$ with $w(S) \geqslant q - w_i$ and $p(S) \geqslant l - p_i$. Its cardinal is given by

$$s_1^i = \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=l-p_i}^{p(N\setminus i)} b_{kr}^i.$$

As $w(S \cup i) \geqslant q$ and $p(S \cup i) \geqslant l$, then s_1^i coincides with the number of winning coalitions in which the player i participates.

On the other hand, inside of the set of the winning coalitions that contain player i, we consider the subset of those coalitions in which player i is not necessary for winning. The cardinal of this subset coincides with the set of all coalitions S such that $i \notin S$ with $w(S) \geqslant q$ and $p(S) \geqslant l$ and it is given by

$$s_2^i = \sum_{k=q}^{w(N\setminus i)} \sum_{r=l}^{p(N\setminus i)} b_{kr}^i.$$

Therefore, the number of swings for player i is $\eta_i(v) = s_1^i - s_2^i$. $\square$

We now establish a generating function to obtain the numbers $\{b_{kr}^i\}_{k,r \geqslant 0}$.

Proposition 2. *Let (N, v) be a weighted double majority game with $v = v_1 \wedge v_2$, where $v_1 = [q; w_1, \ldots, w_n]$ and $v_2 = [l; p_1, \ldots, p_n]$. Then, for each $i \in N$, the generating function of $\{b_{kr}^i\}_{k,r \geqslant 0}$, where b_{kr}^i is the number of coalitions $S \subseteq N$ such that $i \notin S$, $w(S) = k$ and $p(S) = r$ is given by*

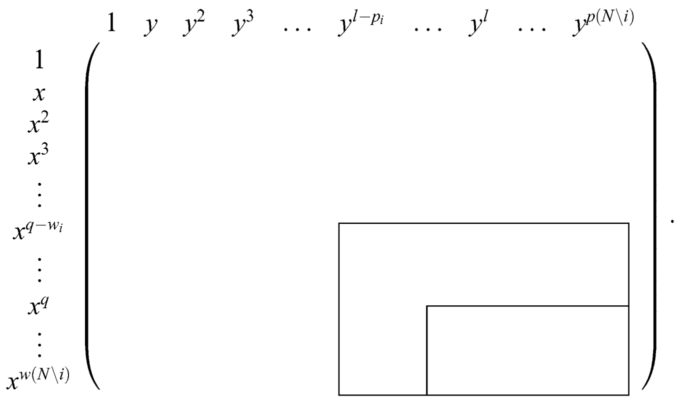
$$B_i(x, y) = \prod_{j=1, j \neq i}^{n} (1 + x^{w_j} y^{p_j}).$$

Proof. Let us consider the polynomial $B(x, y) = \prod_{j=1}^{n} (1 + x^{w_j} y^{p_j})$. Expanding it, we have

$$B(x, y) = \sum_{S \subseteq N} \prod_{i \in S} x^{w_i} y^{p_i} = \sum_{S \subseteq N} x^{w(S)} y^{p(S)} = \sum_{k=0}^{w(N)} \sum_{r=0}^{p(N)} b_{kr} x^k y^r.$$

Therefore, $B(x, y)$ is a generating function for the numbers $\{b_{kr}\}_{k,r \geqslant 0}$ where each b_{kr} is the number of coalitions $S \subseteq N$ such that $w(S) = k$ and $p(S) = r$. Finally, to obtain the numbers $\{b_{kr}^i\}_{k \geqslant 0, r \geqslant 0}$, it suffices to remove the factor $(1 + x^{w_i} y^{p_i})$ in the polynomial $B(x, y)$ giving rise to the generating function $B_i(x, y)$. $\square$

With the aim of making easy the computation of the coefficients $\{b_{kr}^i\}_{k,r \geqslant 0}$, we can use a matrix to store the coefficients of $B_i(x, y)$ (see [4]). If we arrange the rows by increasing powers of x and the columns by increasing powers of y, the element b_{kr}^i is in the position $(k + 1, r + 1)$.



The number of swings of player i is given by the difference between the sums of all elements of the outstanding submatrices. Indeed, the cardinal of the set of winning coalitions that contain player i,

$$s_1^i = \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=l-p_i}^{p(N\setminus i)} b_{kr}^i$$

can be obtained by adding all nonzero elements from the row $q - w_i + 1$ to the row $w(N \setminus i) + 1$, and from the column $l - p_i + 1$ to the column $p(N \setminus i) + 1$. The cardinal of the subset of winning coalitions that contain i in which his presence is not necessary for winning, that is

$$s_2^i = \sum_{k=q}^{w(N\setminus i)} \sum_{r=l}^{p(N\setminus i)} b_{kr}^i$$

can be obtained by adding all nonzero elements from the row $q + 1$ to the row $w(N \setminus i) + 1$, and from the column $l + 1$ to the column $p(N \setminus i) + 1$.

Example 2. We consider the weighted double majority game given by $v = v_1 \wedge v_2$, where $v_1 = [8; 5, 3, 2, 2]$ and $v_2 = [3; 1, 1, 1, 1]$. Its characteristic function is

$$(v_1 \wedge v_2)(S) = \begin{cases} 1 & \text{if } w(S) \geqslant 8 \quad \text{and } p(S) \geqslant 3, \\ 0 & \text{otherwise.} \end{cases}$$

We first calculate the functions $B_i(x, y) = \prod_{j=1, j\neq i}^{n} (1 + x^{w_j} y^{p_j})$.

$$B_1(x, y) = 1 + 2x^2y + x^3y + x^4y^2 + 2x^5y^2 + x^7y^3,$$

$$B_2(x, y) = 1 + 2x^2y + x^5y + x^4y^2 + 2x^7y^2 + x^9y^3,$$

$$B_3(x, y) = 1 + x^2y + x^3y + x^5y + x^5y^2 + x^7y^2 + x^8y^2 + x^{10}y^3,$$

$$B_4(x, y) = 1 + x^2y + x^3y + x^5y + x^5y^2 + x^7y^2 + x^8y^2 + x^{10}y^3.$$

To compute the number of swings for each player the following differences are calculated:

$$\eta_1(v) = \sum_{k=3}^{7} \sum_{r=2}^{3} b_{kr}^1 - \sum_{k=8}^{7} \sum_{r=3}^{3} b_{kr}^1 = 4 - 0 = 4,$$

$$\eta_2(v) = \sum_{k=5}^{9} \sum_{r=2}^{3} b_{kr}^2 - \sum_{k=8}^{9} \sum_{r=3}^{3} b_{kr}^2 = 3 - 1 = 2,$$

$$\eta_3(v) = \sum_{k=6}^{10} \sum_{r=2}^{3} b_{kr}^3 - \sum_{k=8}^{10} \sum_{r=3}^{3} b_{kr}^3 = 3 - 1 = 2,$$

$$\eta_4(v) = \sum_{k=6}^{10} \sum_{r=2}^{3} b_{kr}^4 - \sum_{k=8}^{10} \sum_{r=3}^{3} b_{kr}^4 = 3 - 1 = 2.$$

As the total number of swings is $\overline{\eta}(v) = 10$, the normalized Banzhaf index is

$$\beta(v) = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

For the calculation of the coefficients $\{b_{kr}^i\}_{k,r \geqslant 0}$, we can use a matrix to store the coefficients of $B_i(x, y)$. The matrix that contains the coefficients of $B_3(x, y)$ is

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \\ x^9 \\ x^{10} \end{array} \begin{array}{c} \begin{array}{cccc} 1 & y & y^2 & y^3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Proposition 3. *Let (N, v) be a weighted double majority game with $v = v_1 \wedge v_2$, where $v_1 = [q; w_1, \ldots, w_n]$ and $v_2 = [l; p_1, \ldots, p_n]$. Then,*

(*a*) *the number c of terms of $B(x, y) = \prod_{j=1}^{n} (1 + x^{w_j} y^{p_j})$ verifies*

$$n + 1 \leqslant c \leqslant \min(2^n, w(N)p(N) + 1),$$

(*b*) *the number of terms of $B_i(x, y) = \prod_{j=1, j \neq i}^{n} (1 + x^{w_j} y^{p_j})$, for every $i \in N$, is bounded by c.*

Proof. (a) If the weights of all players are equal, that is $w_i = w$ and $p_i = u$ for all $1 \leqslant i \leqslant n$, then the number of terms of $B(x, y) = (1 + x^w y^u)^n$ is $n + 1$. If the weights are not equal then we obtain more different terms and hence $n + 1$ is less than or equal to the number c of terms of $B(x, y) = \prod_{j=1}^{n} (1 + x^{w_j} y^{p_j})$. On the other hand, we have that

$$B(x, y) = \prod_{j=1}^{n} (1 + x^{w_j} y^{p_j}) = \sum_{k=0}^{w(N)} \sum_{r=0}^{p(N)} b_{kr} x^k y^r$$

is a polynomial of degree $w(N)$ in x, degree $p(N)$ in y, and in which there are no terms such as x^k or y^r. Therefore, $c \leqslant w(N)p(N) + 1$. Moreover, at worst, $c \leqslant 2^n$ because all exponents of the terms of $B(x, y)$ are different and then the number c coincides with the number of the subsets of N.

(b) It follows from (a). □

The *time complexity* function $f : \mathbb{N} \to \mathbb{N}$ of an algorithm give us the maximum time $f(n)$ needed to solve any problem instance of encoding length at most $n \in \mathbb{N}$. A function $f(n)$ is $O(g(n))$ if there is a constant k such that $|f(n)| \leqslant k|g(n)|$ for all integers $n \in \mathbb{N}$. We analyze our algorithms in the *arithmetic model*, that is, we count elementary arithmetic operations and assignments. For instance, the algorithm for computing the product of two $n \times n$ matrices is $O(n^3)$.

The programs of our algorithms contain only *assignments* and **for**-*loop constructs*. We use the symbol $\leftarrow$ for assignments, for example, $g(x) \leftarrow 1$ denotes setting the value of $g(x)$ to 1. A **for**-loop to calculate $\sum_{i=1}^{n} a_i$, can be defined by

$h \leftarrow 0$
for $i \in \{1, \ldots, n\}$ **do**
$h \leftarrow h + a_i$
endfor

output h

Computing the Banzhaf index can be considered as a counting problem: count the number $\eta_i(v)$ of swings for i in game v. This problem in the class of weighted majority games is #P-complete, that is, as hard any counting problem in NP (see [6]). We use generating functions to obtain *pseudo polynomial* algorithms, i.e. polynomial in n and c, for computing the Banzhaf and Shapley–Shubik indices.

Proposition 4. *Let (N, v) be a weighted double majority game with $v = v_1 \wedge v_2$ where $v_1 = [q; w_1, \ldots, w_n]$ and $v_2 = [l; p_1, \ldots, p_n]$. Then,*
(a) expanding out the polynomial $B(x,y) = \prod_{j=1}^{n}(1 + x^{w_j}y^{p_j})$ requires a time $\mathrm{O}(nC)$, where $C = \min(2^n, w(N)\,p(N) + 1)$,
(b) expanding out the polynomial $B_i(x,y) = \prod_{j=1, j\neq i}^{n}(1 + x^{w_j}y^{p_j})$ for every $i \in N$, requires a time $\mathrm{O}(nc)$, where c is the number of terms of $B(x,y)$.

Proof. (a) If $f(n)$ is the number of necessary operations to expand the polynomial

$$B^{(n)}(x,y) = \prod_{j=1}^{n}(1 + x^{w_j}y^{p_j}),$$

we can establish the following recurrence relation:

$$\mathrm{O}(f(n)) = \begin{cases} \mathrm{O}(1) & \text{if } n = 1, \\ \mathrm{O}(f(n-1) + 3C) & \text{if } n \geqslant 2, \end{cases}$$

since $f(1) = 1$ and for $n \geqslant 2$, at worst, the computation of

$$B^{(n)}(x,y) = B^{(n-1)}(x,y)(1 + x^{w_n}y^{p_n})$$

requires a number of products and sums with upper bounds of $2C$ and C, respectively, because C is a upper bound of the number of terms of $B(x,y)$. If we leave out the notation $\mathrm{O}(\cdot)$ and expand the above recurrence, we have

$$f(n) = f(n-1) + 3C = f(n-2) + 2(3C) = \cdots = f(n-k) + k(3C).$$

For $k = n-1$, it holds $f(n) = f(1) + (n-1)(3C)$. Therefore, $\mathrm{O}(f(n)) = \mathrm{O}(nC)$.
(b) It follows from (a). □

Next, we describe the algorithm *BanzTwoPower* which will be used to compute the normalized Banzhaf index of all players in a weighted double majority game and we study its time complexity.

Algorithm. *BanzTwoPower*$(\{w_1, \ldots, w_n\}, \{p_1, \ldots, p_n\}, q, l)$
for $i \in \{1, \ldots, n\}$ **do**

$$B_i(x,y) \leftarrow \prod_{j=1, j\neq i}^{n}(1 + x^{w_j}y^{p_j}),$$

$$B_i(x,y) = \sum_{k=0}^{w(N\setminus i)} \sum_{r=0}^{p(N\setminus i)} b^i_{kr} x^k y^r,$$

$$s^i_1 \leftarrow \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=l-p_i}^{p(N\setminus i)} b^i_{kr},$$

$$s_2^i \leftarrow \sum_{k=q}^{w(N\backslash i)} \sum_{r=l}^{p(N\backslash i)} b_{kr}^i,$$

$\eta_i \leftarrow s_1^i - s_2^i$

endfor
$\overline{\eta} \leftarrow \sum_{i=1}^n \eta_i$
for $i \in \{1, \dots, n\}$ **do**

$\beta_i \leftarrow \frac{\eta_i}{\overline{\eta}}$

endfor
output $\{\beta_1, \dots, \beta_n\}$

Proposition 5. *Let (N, v) be a weighted double majority game with $v = v_1 \wedge v_2$, where $v_1 = [q; w_1, \dots, w_n]$ and $v_2 = [l; p_1, \dots, p_n]$. Computing the normalized Banzhaf index of all players, with the algorithm BanzTwo-Power, requires a time $O(n^2c)$, where c is the number of terms of $B(x, y)$.*

Proof. The time complexity of *BanzTwoPower* is the maximum of the time complexity corresponding to the first loop that determines the number of swings of all players, the time complexity of the calculation of the total number of swings, and the time complexity of the second loop that establishes the Banzhaf power index.

According to Proposition 4, the expansion of each polynomial $B_i(x, y)$ requires a time $O(nc)$ and the storage of the coefficients (b_{kr}^i) requires a time $O(c)$ because c is a upper bound of the number of terms of $B_i(x, y)$. To compute s_1^i and s_2^i, a time $O(c)$ is required and to evaluate $s_1^i - s_2^i$, a time $O(1)$ is necessary. Thus, the time complexity of the first loop is $O(n^2c)$. To compute $\overline{\eta}$ and the second loop, a time $O(n)$ is required. Therefore, the time complexity is $O(n^2c)$. □

For a weighted triple majority game the number of swings of player i is

$$\eta_i(v) = \sum_{k_1=q^1-w_i^1}^{w^1(N\backslash i)} \sum_{k_2=q^2-w_i^2}^{w^2(N\backslash i)} \sum_{k_3=q^3-w_i^3}^{w^3(N\backslash i)} b_{k_1k_2k_3}^i - \sum_{k_1=q^1}^{w^1(N\backslash i)} \sum_{k_2=q^2}^{w^2(N\backslash i)} \sum_{k_3=q^3}^{w^3(N\backslash i)} b_{k_1k_2k_3}^i,$$

where $b_{k_1k_2k_3}^i$ is the number of coalitions S that do not include player i such that $w^t(S) = k_t$, $t = 1, 2, 3$. In this case, we can obtain the numbers $\left\{b_{k_1k_2k_3}^i\right\}_{k_t \geqslant 0}$ for each player $i \in N$, from the generating function

$$B_i(x, y, z) = \prod_{j=1, j\neq i}^{n} (1 + x^{w_j^1} y^{w_j^2} z^{w_j^3}) = \sum_{k_1=0}^{w^1(N\backslash i)} \sum_{k_2=0}^{w^2(N\backslash i)} \sum_{k_3=0}^{w^3(N\backslash i)} b_{k_1k_2k_3}^i x^{k_1} y^{k_2} z^{k_3}.$$

The number c of terms of the polynomial

$$B(x, y, z) = \prod_{j=1}^{n} \left(1 + x^{w_j^1} y^{w_j^2} z^{w_j^3}\right)$$

is bounded in the following way:

$$n + 1 \leqslant c \leqslant \min\left(2^n, \prod_{t=1}^{3} w^t(N) + 1\right).$$

To compute the normalized Banzhaf index in a weighted triple majority game, a time $O(n^2c)$ is required, where c is the number of terms of the generating function $B(x,y,z)$.

4. The Shapley–Shubik index

Let (N,v) be a simple game. The *Shapley–Shubik index* for the player $i \in N$ is defined by

$$\Phi_i(v) = \sum_{\{S \notin \mathscr{W} : S \cup i \in \mathscr{W}\}} \frac{s!(n-s-1)!}{n!},$$

where $n = |N|$, $s = |S|$. If we assume that all $n!$ orderings are equiprobable, then $\Phi_i(v)$ is the probability that player i being the pivotal member of a winning coalition, that is, $S \cup i$ is winning and S is losing. Moreover, for every player i we obtain

$$\Phi_i(v) = \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} d_j^i,$$

where each d_j^i is the number of swings of player i in coalitions of size j.

Similar to what happens with the Banzhaf index in weighted double majority games, it is also possible to obtain, using generating functions, an analogous result for the calculation of the Shapley–Shubik power index. When the game (N,v) is given by $v = v_1 \wedge v_2$, where $v_1 = [q; w_1, \ldots, w_n]$ and $v_2 = [l; p_1, \ldots, p_n]$, then we have that

$$d_j^i = \sum_{k=q-w_i}^{w(N \setminus i)} \sum_{r=l-p_i}^{p(N \setminus i)} a_{krj}^i - \sum_{k=q}^{w(N \setminus i)} \sum_{r=l}^{p(N \setminus i)} a_{krj}^i,$$

where a_{krj}^i is the number of coalitions S such that $i \notin S$ with $w(S) = k$ and $p(S) = r$.

Proposition 6. *Let (N,v) be a weighted double majority game with $v = v_1 \wedge v_2$, where $v_1 = [q; w_1, \ldots, w_n]$ and $v_2 = [l; p_1, \ldots, p_n]$. Then, for each $i \in N$, the generating function of $\{a_{krj}^i\}_{k,r,j \geqslant 0}$, where a_{krj}^i is the number of coalitions S of j players with $w(S) = k$, $p(S) = r$ and such that $i \notin S$, is given by*

$$S_i(x,y,z) = \prod_{j=1, j \neq i}^{n} (1 + x^{w_j} y^{p_j} z).$$

Proof. We consider $S(x,y,z) = (1 + x^{w_1} y^{p_1} z) \cdots (1 + x^{w_n} y^{p_n} z)$. Expanding it, we have

$$S(x,y,z) = \sum_{S \subseteq N} z^{|S|} \prod_{i \in S} x^{w_i} y^{p_i} = \sum_{S \subseteq N} x^{w(S)} y^{p(S)} z^{|S|} = \sum_{k=0}^{w(N)} \sum_{r=0}^{p(N)} \sum_{j=0}^{n} a_{krj} x^k y^r z^j,$$

where each coefficient a_{krj} is the number of coalitions $S \subseteq N$, such that $|S| = j$, $w(S) = k$ and $p(S) = r$. Obviously, to obtain the numbers $\{a_{krj}^i\}_{k,r,j \geqslant 0}$, it suffices to delete the factor $(1 + x^{w_i} y^{p_i} z)$ in the polynomial $S(x,y,z)$ giving rise to their generating function $S_i(x,y,z)$. □

Note that if we know the coefficients $\{a_{krj}^i\}_{k,r,j \geqslant 0}$, using the polynomial $S_i(x,y,z)$, then the numbers $\{d_j^i\}_{j \geqslant 0}$ can be determined. These numbers can be identified with the coefficients of a polynomial $g_i(z) = \sum_{j=0}^{n-1} d_j^i z^j$ and taking into account that

$$d_j^i = \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=l-p_i}^{p(N\setminus i)} a_{krj}^i - \sum_{k=q}^{w(N\setminus i)} \sum_{r=l}^{p(N\setminus i)} a_{krj}^i,$$

it holds

$$g_i(z) = \sum_{j=0}^{n-1} d_j^i z^j = \sum_{j=0}^{n-1} \left(\sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=l-p_i}^{p(N\setminus i)} a_{krj}^i - \sum_{k=q}^{w(N\setminus i)} \sum_{r=l}^{p(N\setminus i)} a_{krj}^i \right) z^j.$$

Hence we obtain that

$$g_i(z) = \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=l-p_i}^{p(N\setminus i)} \left(\sum_{j=0}^{n-1} a_{krj}^i z^j \right) - \sum_{k=q}^{w(N\setminus i)} \sum_{r=l}^{p(N\setminus i)} \left(\sum_{j=0}^{n-1} a_{krj}^i z^j \right),$$

and by Proposition 6 we have that

$$S_i(x,y,z) = \sum_{k=0}^{w(N\setminus i)} \sum_{r=0}^{p(N\setminus i)} \left(\sum_{j=0}^{n-1} a_{krj}^i z^j \right) x^k y^r.$$

The elements of $S_i(x,y,z)$ can be stored in a $(w(N \setminus i)+1) \times (p(N \setminus i)+1)$ matrix where the element of the row $k+1$ and the column $r+1$ is $\sum_{j=0}^{n-1} a_{krj}^i z^j$,

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ \vdots \\ x^k \\ \vdots \\ x^{w(N\setminus i)} \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & y & y^2 & \cdots & y^r & \cdots & y^{p(N\setminus i)} \end{array} \\ \left(\begin{array}{ccccccc} & & & & \vdots & & \\ & & & & \vdots & & \\ & & & & \vdots & & \\ & & & & & & \\ \cdots & \cdots & \cdots & \cdots & \sum_{j=0}^{n-1} a_{krj}^i z^j & & \\ & & & & & & \\ & & & & & & \end{array} \right) \end{array}.$$

Analogous to the calculation of the Banzhaf index, we determine a polynomial in z which coefficients represent the winning coalitions that contain player i,

$$s_1^i(z) = \sum_{k=q-w_i}^{w(N\setminus i)} \sum_{r=l-p_i}^{p(N\setminus i)} \sum_{j=0}^{n-1} a_{krj}^i z^j.$$

On the other hand, we consider a polynomial in z which coefficients represent the number of winning coalitions that contain player i but his presence is not necessary for winning,

$$s_2^i(z) = \sum_{k=q}^{w(N\setminus i)} \sum_{r=l}^{p(N\setminus i)} \sum_{j=0}^{n-1} a_{krj}^i z^j.$$

These polynomials $s_1^i(z)$ and $s_2^i(z)$ are obtained by adding, respectively, the nonzero elements from the row $q - w_i + 1$ to the last and from the column $l - p_i + 1$ to the last, and the nonzero elements from the $w(N \setminus i) - q + 1$ last rows and the $p(N \setminus i) - l + 1$ last columns. Finally,

$$g_i(z) = \sum_{j=0}^{n-1} d_j^i z = s_1^i(z) - s_2^i(z).$$

Proposition 7. *Let* (N, v) *be a weighted double majority game with* $v = v_1 \wedge v_2$ *where* $v_1 = [q; w_1, \ldots, w_n]$ *and* $v_2 = [l; p_1, \ldots, p_n]$. *Then,*

(a) *the number* c *of terms of* $S(x, y, z) = \prod_{j=1}^{n} (1 + x^{w_j} y^{p_j} z)$ *verifies*

$$n + 1 \leqslant c \leqslant \min(2^n, nw(N)p(N) + 1),$$

(b) *the number of terms of* $S_i(x, y, z) = \prod_{j=1, j \neq i}^{n} (1 + x^{w_j} y^{p_j} z)$, *for every* $i \in N$, *is bounded by* c.

Proof. (a) A lower bound is obtained in the case in which the weights of all players are equal, that is, $w_i = w$ and $p_i = u$, for all $1 \leqslant i \leqslant n$. The number of terms of the polynomial $(1 + x^w y^u z)^n$ is always less than or equal to the number of terms of $S(x, y, z) = \prod_{j=1}^{n} (1 + x^{w_j} y^{p_j} z)$. Therefore, $c \geqslant n + 1$. To determine an upper bound we consider that

$$S(x, y, z) = \prod_{j=1}^{n} (1 + x^{w_j} y^{p_j} z) = \sum_{k=0}^{w(N)} \sum_{r=0}^{p(N)} \sum_{j=0}^{n} a_{krj} x^k y^r z^j,$$

is a polynomial of degree $w(N)$ in x, degree $p(N)$ in y, degree n in z, and in which there are no terms such as x^k, y^r or z^j. Therefore, $c \leqslant nw(N)p(N) + 1$. Moreover, at worst, $c \leqslant 2^n$ because all exponents of the $S(x, y, z)$ terms are different and, then c coincides with the number of subsets of N.

(b) It follows from (a). □

The time complexity of the algorithm to expand $S(x, y, z)$ can be obtained by a similar proof to the one that is used in Proposition 4.

Proposition 8. *Let* (N, v) *be a weighted double majority game with* $v = v_1 \wedge v_2$ *where* $v_1 = [q; w_1, \ldots, w_n]$ *and* $v_2 = [l; p_1, \ldots, p_n]$. *Then,*

(a) *expanding out* $S(x, y, z) = \prod_{j=1}^{n} (1 + x^{w_j} y^{p_j} z)$ *requires a time* $O(nC)$, *where*

$$C = \min(2^n, n\,w(N)\,p(N) + 1),$$

(b) *expanding out* $S_i(x, y, z) = \prod_{j=1, j \neq i}^{n} (1 + x^{w_j} y^{p_j} z)$ *for every* $i \in N$, *requires a time* $O(nc)$, *where* c *is the number of terms of* $S(x, y, z)$.

Next we describe the algorithm *ShTwoPower* which will be used to compute the Shapley–Shubik index of all players in a weighted double majority game and we study its time complexity.

Algorithm. *ShTwoPower*$(\{w_1, \ldots, w_n\}, \{p_1, \ldots, p_n\}, q, l)$

for $i \in \{1, \ldots, n\}$ **do**

$$S_i(x, y, z) \leftarrow \prod_{j=1, j \neq i}^{n} (1 + x^{w_j} y^{p_j} z)$$

$$S_i(x,y,z) = \sum_{k=0}^{w(N\backslash i)} \sum_{r=0}^{p(N\backslash i)} \sum_{j=0}^{n-1} a^i_{krj} x^k y^r z^j$$

$$s^i_1(z) \leftarrow \sum_{k=q-w_i}^{w(N\backslash i)} \sum_{r=l-p_i}^{p(N\backslash i)} \sum_{j=0}^{n-1} a^i_{krj} z^j$$

$$s^i_2(z) \leftarrow \sum_{k=q}^{w(N\backslash i)} \sum_{r=l}^{p(N\backslash i)} \sum_{j=0}^{n-1} a^i_{krj} z^j$$

$$g_i(z) \leftarrow s^i_1(z) - s^i_2(z) = \sum_{j=0}^{n-1} d^i_j z$$

$$\Phi_i \leftarrow \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} d^i_j$$

endfor
output $\{\Phi_1, \ldots, \Phi_n\}$

Proposition 9. *Let (N,v) be a weighted double majority game with $v = v_1 \wedge v_2$, where $v_1 = [q; w_1, \ldots, w_n]$ and $v_2 = [l; p_1, \ldots, p_n]$. Computing the Shapley–Shubik power index of all players with the algorithm ShTwoPower requires a time $O(n^2c)$, where c is the number of terms of $S(x,y,z)$.*

Proof. According to Proposition 8, the expansion of each polynomial $S_i(x,y,z)$ requires a time $O(nc)$ and the storage of the coefficients (a^i_{krj}) requires a time $O(c)$ because c is a upper bound of the number of terms of $S_i(x,y,z)$. To compute $s^i_1(z)$ and $s^i_2(z)$, a time $O(c)$ is required and to evaluate $g_i(z) = s^i_1(z) - s^i_2(z)$ a time $O(n)$ is necessary. Thus, the time complexity of the loop is $O(n^2c)$.

To compute Φ_i, a time $O(n^2)$ is required because the factorials are evaluated in time $O(n)$. Therefore, the time complexity of *ShTwoPower* is $O(n^2c)$. □

5. Application to the European Union enlargement

The Council of Ministers of the European Union represents the national governments of the member states. The Council uses a voting system of qualified majority to pass new legislation. The Nice European Council on December 2000 established two decision rules for the European Union enlarged to 27 countries. These rules are contained in the *Declaration on the enlargement of the European Union* and the *Declaration on the qualified majority threshold and the number of votes for a blocking minority in an enlarged Union* (Official Journal of the European Communities 10.3.2001, C 80/80-85).

We next present some results concerning to the Banzhaf and the Shapley–Shubik power indices using the algorithms and results of the previous sections. We compute these indices under the two decision rules prescribed by the Treaty of Nice. Each member state represented in the future Council is considered an individual player. The players in the Council of the European Union enlarged to 27 countries are:

{Germany, United Kingdom, France, Italy, Spain, Poland, Romania, The Netherlands, Greece, Czech Republic, Belgium, Hungary, Portugal, Sweden, Bulgaria, Austria, Slovak Republic, Denmark, Finland, Ireland, Lithuania, Latvia, Slovenia, Estonia, Cyprus, Luxembourg, Malta}.

The first decision rule is the weighted triple majority game $v_1 \wedge v_2 \wedge v_3$, where the three weighted voting games corresponding to votes, countries and population, are the following:

$$v_1 = [255; 29, 29, 29, 29, 27, 27, 14, 13, 12, 12, 12, 12, 12, 10, 10, 10, 7, 7, 7, 7, 7, 4, 4, 4, 4, 4, 3],$$

$$v_2 = [14; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1],$$

$$v_3 = [620; 170, 123, 122, 120, 82, 80, 47, 33, 22, 21, 21, 21, 21, 18, 17, 17, 11, 11, 11, 8, 8, 5, 4, 3, 2, 1, 1].$$

The game v_3 is defined assigning to each country, a number of votes equal to the rate per thousand of its population over the total population and the quota represents the 62% of the total population. So, a voting will be favorable if it counts on the support of 14 countries with at least 255 votes, and at least the 62% of the population.

The second decision rule is the weighted triple majority game $v_1 \wedge v_2' \wedge v_3$, where the weighted voting game v_2' consists of a qualified majority of 2/3 of the countries, that is,

$$v_2' = [18; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].$$

Table 1
The Banzhaf index under the first rule

Countries	Population	Votes	*Pop. I*	*Vote I*	*Game1*	*Game3a*
Germany	82.038	29	0.170	0.084	0.0778	0.0778
United Kingdom	59.247	29	0.123	0.084	0.0778	0.0778
France	58.966	29	0.123	0.084	0.0778	0.0778
Italy	57.612	29	0.120	0.084	0.0778	0.0778
Spain	39.394	27	0.082	0.078	0.0742	0.0742
Poland	38.667	27	0.080	0.078	0.0742	0.0742
Romania	22.489	14	0.047	0.041	0.0426	0.0426
The Netherlands	15.760	13	0.033	0.038	0.0397	0.0397
Greece	10.533	12	0.022	0.035	0.0368	0.0368
Czech Republic	10.290	12	0.021	0.035	0.0368	0.0368
Belgium	10.213	12	0.021	0.035	0.0368	0.0368
Hungary	10.092	12	0.021	0.035	0.0368	0.0368
Portugal	9.980	12	0.021	0.035	0.0368	0.0368
Sweden	8.854	10	0.018	0.029	0.0309	0.0309
Bulgaria	8.230	10	0.017	0.029	0.0309	0.0309
Austria	8.082	10	0.017	0.029	0.0309	0.0309
Slovak Republic	5.393	7	0.011	0.020	0.0218	0.0218
Denmark	5.313	7	0.011	0.020	0.0218	0.0218
Finland	5.160	7	0.011	0.020	0.0218	0.0218
Ireland	3.744	7	0.008	0.020	0.0218	0.0218
Lithuania	3.701	7	0.008	0.020	0.0218	0.0218
Latvia	2.439	4	0.005	0.012	0.0125	0.0125
Slovenia	1.978	4	0.004	0.012	0.0125	0.0125
Estonia	1.446	4	0.003	0.012	0.0125	0.0125
Cyprus	0.752	4	0.002	0.012	0.0125	0.0125
Luxembourg	0.429	4	0.001	0.012	0.0125	0.0125
Malta	0.379	3	0.001	0.009	0.0094	0.0094

Next, we give a Table 1 which contains the Banzhaf index if the first decision rule is used. In the column called *Pop. I* is included the percentage of population over the total, and in the column *Vote I*, the percentage of votes of each country over the total. The Banzhaf index of the game v_1 is included in the column *Game1*, and the Banzhaf index of the weighted triple majority game $v_1 \wedge v_2 \wedge v_3$ is indicated in the column *Game3a*.

Table 2 contains the Shapley–Shubik index for the game $v_1 \wedge v_2 \wedge v_3$, labeled *Game3a*, corresponding to the first decision rule. Note that the Shapley–Shubik indices for the games v_1 and $v_1 \wedge v_2$, labeled *Game1* and *Game2a*, respectively, are equal and the results corresponding to the games $v_1 \wedge v_2 \wedge v_3$ and $v_1 \wedge v_2$ are almost the same.

Next, we include the computations of the Banzhaf and Shapley–Shubik indices for the game $v_1 \wedge v_2' \wedge v_3$, labeled *Game3b*, corresponding to the second decision rule (Table 3). In a similar way, in both cases, these power indices for the game v_1 and the game $v_1 \wedge v_2'$, labeled *Game2b* are compared. So, such as we have already indicated, the results corresponding to the games $v_1 \wedge v_2' \wedge v_3$ and $v_1 \wedge v_2'$ are almost the same.

Summarizing, we obtain the following conclusions:

- The first decision rule which consists of a triple majority system is quasi equivalent to the first game of qualified majority. The power of all countries is almost the same in the games $v_1, v_1 \wedge v_2$, and $v_1 \wedge v_2 \wedge v_3$.
- The second decision rule which differs from the first one because it requires the approval at least of 2/3 of the countries is quasi equivalent to the weighted double majority game $v_1 \wedge v_2'$. In this rule, the required population quota to adopt a decision does not change the power of the countries.

Table 2
The Shapley–Shubik index under the first rule

Countries	Population	*Game1*	*Game2a*	*Game3a*
Germany	0.170	0.0867	0.0867	0.0871
United Kingdom	0.123	0.0867	0.0867	0.0870
France	0.123	0.0867	0.0867	0.0870
Italy	0.120	0.0867	0.0867	0.0870
Spain	0.082	0.0800	0.0800	0.0799
Poland	0.080	0.0800	0.0800	0.0799
Romania	0.047	0.0399	0.0399	0.0399
The Netherlands	0.033	0.0368	0.0368	0.0368
Greece	0.022	0.0341	0.0341	0.0340
Czech Republic	0.021	0.0341	0.0341	0.0340
Belgium	0.021	0.0341	0.0341	0.0340
Hungary	0.021	0.0341	0.0341	0.0340
Portugal	0.021	0.0341	0.0341	0.0340
Sweden	0.018	0.0282	0.0282	0.0281
Bulgaria	0.017	0.0282	0.0282	0.0281
Austria	0.017	0.0282	0.0282	0.0281
Slovak Republic	0.011	0.0196	0.0196	0.0196
Denmark	0.011	0.0196	0.0196	0.0196
Finland	0.011	0.0196	0.0196	0.0196
Ireland	0.008	0.0196	0.0196	0.0196
Lithuania	0.008	0.0196	0.0196	0.0196
Latvia	0.005	0.0110	0.0110	0.0110
Slovenia	0.004	0.0110	0.0110	0.0110
Estonia	0.003	0.0110	0.0110	0.0110
Cyprus	0.002	0.0110	0.0110	0.0110
Luxembourg	0.001	0.0110	0.0110	0.0110
Malta	0.001	0.0082	0.0082	0.0082

Table 3
The power indices under the second rule

Countries	Banzhaf indices			Shapley–Shubik indices		
	Game1	*Game2b*	*Game3b*	*Game1*	*Game2b*	*Game3b*
Germany	0.0778	0.0665	0.0665	0.0867	0.0834	0.0837
United Kingdom	0.0778	0.0665	0.0665	0.0867	0.0834	0.0836
France	0.0778	0.0665	0.0665	0.0867	0.0834	0.0836
Italy	0.0778	0.0665	0.0665	0.0867	0.0834	0.0836
Spain	0.0742	0.0631	0.0631	0.0800	0.0768	0.0767
Poland	0.0742	0.0631	0.0631	0.0800	0.0768	0.0767
Romania	0.0426	0.0407	0.0407	0.0399	0.0395	0.0394
The Netherlands	0.0397	0.0386	0.0386	0.0368	0.0366	0.0365
Greece	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Czech Republic	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Belgium	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Hungary	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Portugal	0.0368	0.0366	0.0366	0.0341	0.0341	0.0340
Sweden	0.0309	0.0325	0.0325	0.0282	0.0287	0.0286
Bulgaria	0.0309	0.0325	0.0325	0.0282	0.0287	0.0286
Austria	0.0309	0.0325	0.0325	0.0282	0.0287	0.0286
Slovak Republic	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Denmark	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Finland	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Ireland	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Lithuania	0.0218	0.0263	0.0263	0.0196	0.0209	0.0208
Latvia	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Slovenia	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Estonia	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Cyprus	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Luxembourg	0.0125	0.0198	0.0198	0.0110	0.0131	0.0131
Malta	0.0094	0.0177	0.0177	0.0082	0.0106	0.0106

- The two rules of triple majority adopted in the Nice summit meeting are almost equivalent to a simple majority game (the first) or a double majority game (the second). With both rules, the required population quota to adopt a decision does not change in practice the power of the countries.

The algorithms and the power indices obtained for the European Union enlarged to 27 countries, taking into account the Nice weighting of votes, are contained in the *Notebook* of the system Mathematica, due to [2].

Acknowledgements

This research has been partially supported by the Spanish Ministry of Science and Technology, under grant SEC2000–1243.

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