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Note

A note on a value with incomplete communication

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Abstract

The Myerson's models on partial cooperation have been studied extensively [SIAM J. Discrete Math. 5 (1992) 305; Math. Methods Operations Res. 2 (1977) 225; Int. J. Game Theory 19 (1980) 421; 20 (1992) 255]. In [Game Econ. Behav. 26 (1999) 565], Hamiache proposes a new solution concept for communication situations. In this work, we analyze this value making some deficiencies clear and generalize this value to union stable cooperation structures emphasizing the differences in the extension.

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Keywords: Cooperative game; Communication situation; Consistency

1. Introduction

In the general model of cooperative games it is assumed that there is no restriction in the formation of coalitions and so, a *cooperative game* with transferable utility is defined as a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function that assigns to each $S \subseteq N$ a worth $v(S)$ and verifies that $v(\emptyset) = 0$. However, in many practical situations the cooperation is not complete. The *communication situations* model (Myerson, 1977; Owen, 1986), where the relationships among the players are represented by undirected

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graphs, is one of the most important. The restricted game by a graph (N, G) is defined by $v^G : 2^N \to \mathbb{R}$, $v^G(S) = \sum_{T \in S/G} v(T)$ where S/G is the set of connected components of $S \subseteq N$. The most extensively studied solution concepts in communication situations have been the Shapley value of the restricted game or Myerson (1977) value, and the position value (Meesen, 1988; Borm et al., 1992) which discriminates the value of each player by defining a game over the edges of the graph.

In the paper "A Value with Incomplete Communication," Hamiache (1999) makes a critical valuation of the Myerson value and the position value in communication situations and concludes that these values do not discriminate the players enough according to their position in the graph. He proposes a new solution concept for communication situations and gives an axiomatic characterization based mainly on the notions of associated game and consistency.

In this work, we analyze this value and indicate some deficiencies in the results. Moreover, we generalize this value to *union stable cooperation structures,* which have communication situations as a particular case. The justification of union stable structures comes from Myerson himself who pointed out the limitations of communication situations and modeled the relationships among the players by means of hypergraphs (Myerson, 1980). Later, van den Nouweland et al. (1992), Slikker and van den Nouweland (2001) studied these structures through communication hypergraphs.

2. A value with incomplete communication

In this section, we resume the Hamiache's results. We denote a communication situation by (N, v, G) where (N, v) is a cooperative game and (N, G) is a graph and by SC^N the set of all communication situations on N. Given a graph (N, G) and $S \subseteq N$, let $S^* = \{i \in N\colon \exists j \in S \text{ such that } \{i, j\} \in G\}$.

If ϕ is a solution on SC^N, for all (N, v, G), its associated game (N, v^*_ϕ, G) is defined, for $S \subseteq N$, by

$$v^*_\phi(S) = \begin{cases} v(S) + \sum_{j \in S^* \setminus S} \big[\phi_j\big(S \cup \{j\}, v_{S\cup\{j\}}, G(S \cup \{j\})\big) - v(\{j\})\big] \\ \quad \text{if } S \text{ connected}, \\ \sum_{T \in S/G} v^*_\phi(T) \quad \text{otherwise}, \end{cases}$$

where $(T, v_T, G(T))$ is the communication situation restricted to the coalition T.

Hamiache formulates the following axioms, where u_R is the unanimity game corresponding to the coalition R, that is, for $S \subseteq N$,

$$u_R(S) = \begin{cases} 1 & \text{if } R \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

H1 Component-efficiency. For (N, v, G) and $S \in N/G$, $\sum_{j \in S} \phi_j(N, v, G) = v(S)$.

H2 Linearity with respect to the game. For all $\alpha, \beta \in \mathbb{R}$ and (N, v, G), $(N, w, G) \in SC^N$, $\phi(N, \alpha v + \beta w, G) = \alpha\phi(N, v, G) + \beta\phi(N, w, G)$.

H3 Independence of irrelevant players. For all (N, G), for all connected coalitions R, T with $R \subseteq T$ and for $i \in T$, $\phi_i(N, u_R, G) = \phi_i(T, u_R, G(T))$.

H4 Positivity. For all connected coalitions T and for $i \in T$, $\phi_i(T, u_T, G(T)) \geqslant 0$.
H5 Associated consistency. For all (N, v, G), $\phi(N, v, G) = \phi(N, v^*_\phi, G)$.

Using this set of axioms, Hamiache proves the following results.

Lemma 1. *For all connected coalitions $R \subseteq N$,*

$$\phi_i(N, u_R, G) = \begin{cases} \phi_i(R, u_R, G(R)) & \text{if } i \in R, \\ 0 & \text{if } i \notin R. \end{cases}$$

Lemma 2. *For all $(N, v, G) \in SC^N$,*

$$v^G = \sum_{\substack{R \subseteq N \\ R \text{ connected}}} c_R u_R \quad \text{with} \quad c_R = \sum_{\substack{S \text{ connected} \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S} v(S).$$

Theorem 3. *There is only one* (*unique*) *solution ϕ on the set of games with communication structures, $\{(N, v^G, G)\colon (N, v, G) \in SC^N\}$, which satisfies the set of axioms* H1–H5.

Theorem 4. $\phi(N, v, G) = \phi(N, v^G, G)$, *for all $(N, v, G) \in SC^N$.*

In the proofs of these theorems, Hamiache supposes that the graph is connected. He asserts that this assumption can be made because of axioms H1 and H2 and Lemma 1. However, this reasoning is not correct since in the proof of Lemma 1 the connectivity of the graph is assumed and so, Theorem 3 (Hamiache, 1999, Theorem 1, p. 68) is not true. The existence part of Hamiache is correct. If $N/G = \{S_1, \ldots, S_k\}$ is the partition of N into the connected components, for each communication structure $(S_j, G(S_j))$, we can apply Hamiache's argument and obtain the unique value $\phi|_{S_j}$. Then $\phi := (\phi|_{S_1}, \ldots, \phi|_{S_k})$ is the required value on Hamiache's subclass of games. The uniqueness part of Hamiache is false, and here is a counterexample, given a particular graph (N, G).[1] We will make extensive use of the linearity axiom and the formula obtained by Hamiache (1999, Lemma 2, p. 67):

$$\phi_i(N, v, G) = \sum_{\substack{S \subseteq N \\ S \text{ connected} \\ i \in S^*}} \sum_{\substack{R \\ S \subseteq R \subseteq S^* \\ i \in R}} (-1)^{|R|-|S|} \phi_i\big(R, u_R, G(R)\big) v(S).$$

Let $N = \{1, 2, 3\}$ and $G = \{S_1, S_2\}$, $S_1 = \{1\}$, $S_2 = \{2, 3\}$. Let $\phi = (\phi|_{S_1}, \phi|_{S_2})$ be the value obtained in the proof of the existence part of the theorem. Define ψ as follows: Fix any $c \neq 0$, and choose any connected set R and any set T such that $R \subseteq T \subseteq N$.

$$\psi_i\big(T, u_R, G(T)\big) := \begin{cases} \phi_i(T, u_R, G(T)) & \text{if } i \in T \neq N, \\ \phi_i(N, u_R, G) & \text{if } [i = 1 \text{ or } R \neq \{1\}],\ T = N, \\ c & \text{if } [i = 2 \text{ and } R = \{1\}],\ T = N, \\ -c & \text{if } [i = 3 \text{ and } R = \{1\}],\ T = N. \end{cases}$$

[1] We want to thank an anonymous referee for his comments on an earlier draft of this paper and the formulation of this counterexample.

It is easy to check that ψ satisfies the set of axioms H1–H5 for the value defined on the games $\{(T, v_T, G(T))\colon\ T \subseteq N,\ v = v^G\}$. However, when we consider the case of $R = S_1$ and $T = N$, $\psi_2(N, u_{S_1}, G) = c$ and $\Psi_3(N, u_{S_1}, G) = -c$ by definition. Obviously Lemma 1 is violated.

With the aim of repairing this mistake, we have reformed the system of axioms in our generalization of this value for union stable structures in the following section. This generalization is not immediate as the formula established in Lemma 2 of Hamiache's work for graphs is not true in our context.

3. A value with incomplete communication for union stable cooperation structures

First of all, we introduce some basic concepts on union stable structures (see Algaba et al., 2000, 2001).

Definition 5. A union stable system is a pair $(N, \mathcal{F})$ with $\mathcal{F} \subseteq 2^N$ verifying that $\{i\} \in \mathcal{F}$ for all $i \in N$ and for all $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, it holds $S \cup T \in \mathcal{F}$.

Given $(N, \mathcal{F})$ a union stable system, we denote by $\mathcal{B}(\mathcal{F})$ the set of all coalitions which cannot be expressed as a union of feasible coalitions with nonempty intersection. This set $\mathcal{B}(\mathcal{F})$ is called *basis of* $\mathcal{F}$ and its elements are the *supports of* $\mathcal{F}$.

A *union stable cooperation structure* is a triple $(N, v, \mathcal{F})$ where (N, v) is a cooperative game and $(N, \mathcal{F})$ is a union stable system. The set of all union stable cooperation structures with players set N will be denoted by US^N.

Definition 6. Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. The restricted game by $\mathcal{F}$, $v^{\mathcal{F}} : 2^N \to \mathbb{R}$, is defined, for all $S \subseteq N$, by $v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$ where $C_{\mathcal{F}}(S) = \{T \in \mathcal{F}\colon\ T \subseteq S \text{ and } \nexists T' \in \mathcal{F} \text{ such that } T \subsetneq T' \subseteq S\}$.

In the following, let $RUS^N = \{(N, v^{\mathcal{F}}, \mathcal{F})\colon\ (N, v, \mathcal{F}) \in US^N\}$. It is easy to check that $\{u_R\colon\ R \neq \emptyset,\ R \in \mathcal{F}\}$ is a basis of the vector space $\{v^{\mathcal{F}}\colon\ (N, v, \mathcal{F}) \in US^N\}$. Thus,

$$v^{\mathcal{F}}(S) = \sum_{\{R \in \mathcal{F}\colon\ R \neq \emptyset\}} c_R u_R(S) = \sum_{\{R \in \mathcal{F}\colon\ R \subseteq S\}} c_R,$$

for all $S \in \mathcal{F}$, $S \neq \emptyset$, and applying the Möbius Inversion Formula (Stanley, 1986) we obtain

$$c_R = \sum_{\{S \in \mathcal{F}\colon\ S \subseteq R\}} \mu(S, R) v(S)$$

where

$$\mu(S, R) = \begin{cases} 1 & \text{if } S = R, \\ -\sum_{\{H\colon\ S \subseteq H \subsetneq R\}} \mu(S, H) & \text{if } S \subsetneq R. \end{cases} \tag{1}$$

Given $(N, \mathcal{F})$ and $S \in \mathcal{F}$, let $S^* = \{i \in N\colon\ \exists B \in \mathcal{B}(\mathcal{F}) \text{ with } i \in B \text{ and } B \cap S \neq \emptyset\}$. It is evident that $S^* = \bigcup\{B \in \mathcal{B}(\mathcal{F})\colon\ B \cap S \neq \emptyset\}$, and $S \subseteq S^* \in \mathcal{F}$.

Note that if $(N, v, \mathcal{F}) \in SC^N$, the definition of S^* coincides with the Hamiache's since $\mathcal{B}(\mathcal{F})$ would be formed by the edges of the graph. However, our extension of S^* does not allow us to calculate the coefficients c_R by means of Lemma 2 as we show in this example: let $N = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3\}, \{3, 4\}, N\}$ and let v be the game given by $v(S) = |S|^2$ for $S \subseteq N$. If we take $R = \{1, 2, 3\}$ and use the recurrence formula to calculate c_R, that is, $c_R = v(R) - \sum_{\{S\in\mathcal{F}:\ S\subsetneq R\}} c_S$ with $c_\emptyset = 0$, then $c_R = 6$. However, if we use the extended formula in Lemma 2, i.e.,

$$c_R = \sum_{\{S\in\mathcal{F}:\ S\subseteq R\subseteq S^*\}} (-1)^{|R|-|S|} v(S),$$

it holds that $c_R = 12$. For this reason, we use (1) for the coefficients c_R.

Let ϕ be a solution on US^N and let $(N, v, \mathcal{F}) \in US^N$. We define its associated game $(N, v^*_\phi, \mathcal{F})$ by

$$v^*_\phi(S) = \begin{cases} v(S) + \sum_{j\in S^*\setminus S} \frac{\sum_{B\in B(j,S)}[\phi_j(S\cup B, v_{S\cup B}, \mathcal{F}_{S\cup B}) - v(\{j\})]}{|B(j,S)|} & \text{if } S \in \mathcal{F}, \\ \sum_{T\in C_\mathcal{F}(S)} v^*_\phi(T) & \text{if } S \notin \mathcal{F}, \end{cases}$$

where $B(j, S) = \{B \in \mathcal{B}(\mathcal{F}):\ j \in B \text{ and } B \cap S \neq \emptyset\}$. In the following we denote by $(N, v^*, \mathcal{F})$ the associated game with respect to ϕ and by $(T, v_T, \mathcal{F}_T)$ the union stable structure restricted to the coalition T where $\mathcal{F}_T = \{F \in \mathcal{F}:\ F \subseteq T\}$. In the associated game $\phi_j(S \cup B, v_{S\cup B}, \mathcal{F}_{S\cup B}) - v(\{j\})$ represents the surplus of player j when he is invited by S to play the game $(S \cup B, v_{S\cup B}, \mathcal{F}_{S\cup B})$. Since the members of a feasible coalition S can only communicate directly with players $j \in S^* \setminus S$ and each player j can belong to more than one support, an average of contributions of the player to the different coalitions $S \cup B$ is considered. The negotiation is only established with those players that belong to any support B such that $S \cap B \neq \emptyset$. We now consider a set of axioms.

A1 Component-independence. For all $(N, v, \mathcal{F}) \in US^N$ and $i \in N$, $\phi_i(N, v, \mathcal{F}) = \phi_i(M, v_M, \mathcal{F}_M)$, with $M \in C_\mathcal{F}(N)$ such that $i \in M$.

A2 Component-efficiency. For all $(N, v, \mathcal{F}) \in US^N$ and $M \in C_\mathcal{F}(N)$, $\sum_{k\in M} \phi_k(N, v, \mathcal{F}) = v(M)$.

A3 Linearity with respect to the game. For all $\alpha, \beta \in \mathbb{R}$ and $(N, v, \mathcal{F})$, $(N, w, \mathcal{F}) \in US^N$, $\phi(N, \alpha v + \beta w, \mathcal{F}) = \alpha\phi(N, v, \mathcal{F}) + \beta\phi(N, w, \mathcal{F})$.

A4 Independence of irrelevant players. For all $(N, \mathcal{F})$ and $R, T \in \mathcal{F}$ with $R \subseteq T$, $\phi_i(N, u_R, \mathcal{F}) = \phi_i(T, u_R, \mathcal{F}_T)$ for all $i \in T$.

A5 Positivity. For all $R, T \in \mathcal{F}$ with $R \subseteq T$, $\phi_i(T, u_R, \mathcal{F}_T) \geqslant 0$ for $i \in T$ and $\phi_i(R, u_R, \mathcal{F}_R) > 0$ for $i \in R$.

A6 Associated consistency. For all $(N, v, \mathcal{F}) \in US^N$, $\phi(N, v, \mathcal{F}) = \phi(N, v^*_\phi, \mathcal{F})$.

Unlike the set of axioms of Hamiache, we have introduced a new axiom (A1). This axiom indicates that the value of one player in a union stable cooperation structure $(N, v, \mathcal{F})$ only depends on the maximal feasible coalition that contains this player. The introduction of this axiom is logical because the set of the maximal feasible coalitions in $\mathcal{F}$ form a partition of N. Moreover, this axiom is a characteristic property of the classical values in the literature on games with partial cooperation (Algaba et al., 2000, 2001; Myerson, 1977; van den Nouweland et al., 1992; Owen, 1986).

By Axiom 1 we can assume, without loss of generality, that $N \in \mathcal{F}$. Moreover, we establish the following result where our proof is different to Hamiache's proof.

Lemma 7. *If ϕ is a solution on US^N satisfying* A1–A6*, then, for all $R \in \mathcal{F}$,*

$$\phi_i(N, u_R, \mathcal{F}) = \begin{cases} \phi_i(R, u_R, \mathcal{F}_R) & \text{if } i \in R, \\ 0 & \text{if } i \notin R. \end{cases}$$

Proof. Let $R \in \mathcal{F}$. If $i \in R$, the result is immediate by A4. When $i \notin R$, let us consider two cases: $R \subseteq M$ and $R \nsubseteq M$ where $M \in C_{\mathcal{F}}(N)$ such that $i \in M$.

(1) If $R \subseteq M$ then $\sum_{j \in R} \phi_j(N, u_R, \mathcal{F}) = \sum_{j \in R} \phi_j(R, u_R, \mathcal{F}_R) = 1$ by A4 and A2 since $R \in C_{\mathcal{F}_R}(R)$. Also, by A1 and A2,

$$\sum_{j \in M} \phi_j(N, u_R, \mathcal{F}) = \sum_{j \in M} \phi_j(M, u_R, \mathcal{F}_M) = u_R(M) = 1.$$

If we subtract both expressions, then $\sum_{j \in M \setminus R} \phi_j(N, u_R, \mathcal{F}) = 0$, and for $j \in M$, we obtain $\phi_j(N, u_R, \mathcal{F}) = \phi_j(M, u_R, \mathcal{F}_M) \geqslant 0$ by $A5$. Therefore, $\phi_i(N, u_R, \mathcal{F}) = 0$.

(2) If $R \nsubseteq M$ then u_R is the null game on M. Therefore, using the linearity axiom, $\phi_i(N, u_R, \mathcal{F}) = \phi_i(M, u_R, \mathcal{F}_M) = \phi_i(M, 0 \cdot u_R, \mathcal{F}_M) = 0$. □

Let ϕ be a solution on US^N satisfying the axioms A1–A6. Then, for $i \in N$,

$$\begin{aligned} \phi_i\big(N, v^{\mathcal{F}}, \mathcal{F}\big) &= \sum_{\{R \in \mathcal{F}:\ R \neq \emptyset\}} c_R \phi_i(N, u_R, \mathcal{F}) = \sum_{\{R \in \mathcal{F}:\ i \in R\}} c_R \phi_i(R, u_R, \mathcal{F}_R) \\ &= \sum_{\{R \in \mathcal{F}:\ i \in R\}} \Bigg[\sum_{\{S \in \mathcal{F}:\ S \subseteq R\}} \mu(S, R) v(S) \Bigg] \phi_i(R, u_R, \mathcal{F}_R), \end{aligned}$$

and hence

$$\phi_i\big(N, v^{\mathcal{F}}, \mathcal{F}\big) = \sum_{S \in \mathcal{F}} \Bigg[\sum_{\{R \in \mathcal{F}:\ R \supseteq S, i \in R\}} \mu(S, R) \phi_i(R, u_R, \mathcal{F}_R) \Bigg] v(S). \tag{2}$$

The technical difficulties of this expression forces a condition on the union stable systems $(N, \mathcal{F})$:

(C) if $S \in \mathcal{F}$ and there exists $B \in \mathcal{B}(\mathcal{F})$ such that $S \cap B \neq \emptyset$ and $S \cup B = N$, then there exists no $F \in \mathcal{F}$ so that $S \subsetneq F \subsetneq N$.

Theorem 8. *There is only one solution ϕ on the set of union stable cooperation structures $\{(N, v^{\mathcal{F}}, \mathcal{F}) \in RUS^N:\ (N, \mathcal{F})$ verifies* (C)$\}$ *which satisfies the set of axioms* A1–A6.

Proof. Each union stable system $(N, \mathcal{F})$ has a unique basis and so, the elements of RUS^N can be classified by the number of non-singleton supports. We will prove the result by induction on the number of non-singleton supports of $\mathcal{F}$.

If $(N, \mathcal{F})$ is a union stable system (assume $N \in \mathcal{F}$) for which there is no non-singleton support, then $\mathcal{F} = \{N\}$ with $N = \{1\}$ and it is easy to check that there is only one solution ϕ satisfying A1–A6 given by $\phi_1(N, v^{\mathcal{F}}, \mathcal{F}) = v^{\mathcal{F}}(\{1\}) = v(\{1\})$.

Assume that there exists a unique solution ϕ verifying A1–A6 for all union stable systems with $k-1$ non-singleton supports and let $(N,\mathcal{F})$ be a union stable system with k non-singleton supports. By A6, $\phi_i(N,v^{\mathcal{F}},\mathcal{F})=\phi_i(N,(v^{\mathcal{F}})^*,\mathcal{F})$ for all $i\in N$ and taking into account that $v^{\mathcal{F}}=\sum_{\{R\in\mathcal{F}:\ R\neq\emptyset\}}c_R u_R$, then, by application of Lemma 7,

$$\sum_{\{R\in\mathcal{F}:\ i\in R\}} c_R\phi_i(R,u_R,\mathcal{F}_R)=\sum_{\{R\in\mathcal{F}:\ i\in R\}} c_R\phi_i\big(R,u_R^*,\mathcal{F}_R\big).$$

For any $R\subsetneq N$, the number of non-singleton supports of $(R,\mathcal{F}_R)$ is less than the number of non-singleton supports of $(N,\mathcal{F})$. Therefore, by induction hypothesis $\phi_i(N,v^{\mathcal{F}},\mathcal{F})=\phi_i(N,(v^{\mathcal{F}})^*,\mathcal{F})$ if and only if $\phi_i(N,u_N,\mathcal{F})=\phi_i(N,u_N^*,\mathcal{F})$. Using expression (2) and knowing that, for $S\in\mathcal{F}$,

$$u_N^*(S)=u_N(S)+\sum_{j\in S^*\setminus S}\frac{\sum_{B\in B(j,S)}\phi_j(S\cup B,u_N,\mathcal{F}_{S\cup B})}{|B(j,S)|},$$

then $\phi_i(N,u_N,\mathcal{F})=\phi_i(N,u_N^*,\mathcal{F})$ if and only if

$$\sum_{S\in\mathcal{F}}\Bigg[\sum_{\substack{R\in\mathcal{F}\\ R\supseteq S,i\in R}}\mu(S,R)\phi_i(R,u_R,\mathcal{F}_R)\Bigg]\sum_{j\in S^*\setminus S}\frac{\sum_{B\in B(j,S)}\phi_j(S\cup B,u_N,\mathcal{F}_{S\cup B})}{|B(j,S)|}=0.$$

Note that if $S\cup B\subsetneq N$, then u_N restricted to $S\cup B$ is the null function and, by linearity axiom, $\phi_j(S\cup B,u_N,\mathcal{F}_{S\cup B})=0$ for $j\in S\cup B$. Thus, we only consider $S\in\mathcal{F}$ such that $S\cup B=N$ for some $B\in\mathcal{B}(\mathcal{F})$ with $B\cap S\neq\emptyset$. For these coalitions, $S^*=N$. If we denote by $\mathcal{F}(N)=\{S\in\mathcal{F}:\ \exists B\in\mathcal{B}(\mathcal{F})$ with $S\cap B\neq\emptyset,\ S\cup B=N\}$, $\mathcal{F}(i,N)=\{S\in\mathcal{F}(N):\ i\in S\}$ and take into account that $(N,\mathcal{F})$ verifies (C), then for each $S\in\mathcal{F}$ with $S\cup B=N$, $S\cap B\neq\emptyset$, the only coalitions $R\in\mathcal{F}$ with $R\supseteq S$ and $i\in R$ are S and N if $i\in S$ and only the coalition N if $i\notin S$. Therefore, it must be

$$\sum_{S\in\mathcal{F}(i,N)}\big[\mu(S,S)\phi_i(S,u_S,\mathcal{F}_S)+\mu(S,N)\phi_i(N,u_N,\mathcal{F})\big]\sum_{j\in N\setminus S}\phi_j(N,u_N,\mathcal{F})$$
$$+\sum_{\{S\in\mathcal{F}(N):\ i\notin S\}}\big[\mu(S,N)\phi_i(N,u_N,\mathcal{F})\big]\sum_{j\in N\setminus S}\phi_j(N,u_N,\mathcal{F})=0.$$

By definition of Möbius function, $\mu(S,S)=1$ and $\mu(S,N)=-1$. Hence,

$$\sum_{S\in\mathcal{F}(i,N)}\phi_i(S,u_S,\mathcal{F}_S)\sum_{j\in N\setminus S}\phi_j(N,u_N,\mathcal{F})$$
$$=\sum_{S\in\mathcal{F}(N)}\phi_i(N,u_N,\mathcal{F})\sum_{j\in N\setminus S}\phi_j(N,u_N,\mathcal{F}),$$

and by changing the order of summations

$$\sum_{j\in N}\Bigg[\sum_{\substack{S\in\mathcal{F}(i,N)\\ j\in N\setminus S}}\phi_i(S,u_S,\mathcal{F}_S)\Bigg]\phi_j(N,u_N,\mathcal{F})$$
$$=\sum_{S\in\mathcal{F}(N)}\sum_{j\in N\setminus S}\phi_j(N,u_N,\mathcal{F})\phi_i(N,u_N,\mathcal{F}).$$

This equation in matrix form is $M\phi(N,u_N,\mathcal{F})=\lambda\phi(N,u_N,\mathcal{F})$ where $M=[m_{ij}]_{n\times n}$ is the matrix defined by

$$m_{ij}=\begin{cases}\sum_{\{S\in\mathcal{F}(i,N):\ j\in N\setminus S\}}\phi_i(S,u_S,\mathcal{F}_S) & \text{if } \{S\in\mathcal{F}(i,N):\ j\in N\setminus S\}\neq\emptyset,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\lambda=\sum_{S\in\mathcal{F}(N)}\sum_{j\in N\setminus S}\phi_j(N,u_N,\mathcal{F}). \tag{3}$$

Therefore, $\phi(N,u_N,\mathcal{F})$ is our solution if and only if it is a nonnegative eigenvector of the matrix M.

Note that, for any $i,j\in N$ there exists at most one coalition $S\in\mathcal{F}(i,N)$ such that $j\in N\setminus S$. Indeed, we assume that there exist $S_1,S_2\in\mathcal{F}(i,N)$ with $j\in N\setminus S_1$ and $j\in N\setminus S_2$. Since $i\in S_1\cap S_2$ and $j\notin S_1\cup S_2$, $S_1\cup S_2\in\mathcal{F}$ and $S_1\cup S_2\neq N$. Therefore, there exists $T=S_1\cup S_2\in\mathcal{F}$ such that $S_1\subsetneq T\subsetneq N$. This is a contradiction to condition (C) of $(N,\mathcal{F})$. Thus, M can be written as

$$m_{ij}=\begin{cases}\phi_i(S,u_S,\mathcal{F}_S) & \text{if there exists } S\in\mathcal{F}(i,N) \text{ with } j\in N\setminus S,\\ 0 & \text{otherwise.}\end{cases}$$

Let $A_{\mathcal{F}}=\{i\in N:\ i\in S \text{ for all } S\in\mathcal{F}(N)\}$ be the set of all *articulation points* of $\mathcal{F}$. We distinguish two cases:

Case 1. There is only one non-singleton support, $\mathcal{F}=\{\{1\},\dots,\{n\},N\}$. In this case, there is no articulation point,

$$m_{ij}=\begin{cases}1 & \text{if } i\neq j,\\ 0 & \text{if } i=j,\end{cases}\quad\text{and}\quad \lambda=\sum_{i\in N}\sum_{j\in N\setminus\{i\}}\phi_j(N,u_N,\mathcal{F})=n-1.$$

The corresponding eigenvector summing up to one is unique, $\phi_i(N,u_N,\mathcal{F})=1/n$ for all $i\in N$.

Case 2. There is more than one non-singleton supports, $\mathcal{F}$ contains at least one articulation point. The matrix M is nonnegative and the diagonal elements are zero (Bapat and Raghavan, 1997). The columns of M corresponding to articulation points are zero vectors and the sum of the elements of a column corresponding to one no articulation point is 1. The number $\lambda=0$ is an eigenvalue with multiplicity equal to at least the number of articulation points. The corresponding eigenvectors have nonzero components only for the articulation points and so, we must discard them because they do not verify the positivity axiom. The remaining eigenvalues are those of the submatrix $M(N\setminus A_{\mathcal{F}})$ obtained by deleting lines and columns corresponding to articulation points. As $M(N\setminus A_{\mathcal{F}})$ is a nonnegative irreducible matrix, the Perron–Frobenius theorem can be applied to this submatrix, and we can choose the unique eigenvalue $0<\lambda<1$ with multiplicity 1 and its corresponding eigenvector $y>0$ verifying (3). This number λ is eigenvalue of M and its corresponding eigenvector x given by

$$x_i=\begin{cases}y_i & \text{if } i\in N\setminus A_{\mathcal{F}},\\ \frac{1}{\lambda}\sum_{j\in N\setminus A_{\mathcal{F}}}m_{ij}y_j & \text{if } i\in A_{\mathcal{F}},\end{cases}$$

is the only one vector satisfying axioms A1–A6. □

Note that although the proof of Theorem 8 follows the technique of Hamiache, in our generalization the induction has been made on the number of non-singleton supports of $(N,\mathcal{F})$, not on the number of players. However, the matrix obtained in the generalization is the same as Hamiache's when the union stable structure is a communication situation.

Theorem 9. *If ϕ is a solution on $\{(N,v,\mathcal{F})\in US^N\colon (N,\mathcal{F})$ verifies (C)$\}$ which satisfies the set of axioms A1–A6, then $\phi(N,v,\mathcal{F})=\phi(N,v^{\mathcal{F}},\mathcal{F})$.*

Proof. We prove the result by induction on the number of non-singleton supports of $\mathcal{F}$. Obviously, given $(N,v,\mathcal{F})$ with $\mathcal{F}=\{\{1\}\}$, it holds $\phi(N,v,\mathcal{F})=\phi(N,v^*,\mathcal{F})=\phi(N,(v^{\mathcal{F}})^*,\mathcal{F})=\phi(N,v^{\mathcal{F}},\mathcal{F})$.

Assume that the result holds for all union stable systems with $k-1$ non-singleton supports and let $(N,\mathcal{F})$ be a union stable system with k non-singleton supports. If $w=(v^{\mathcal{F}})^*-v^*$, then,

$$\phi_i(N,w,\mathcal{F})=\sum_{S\in\mathcal{F}}\left[\sum_{\{R\in\mathcal{F}\colon R\supseteq S,i\in R\}}\mu(S,R)\phi_i(R,u_R,\mathcal{F}_R)\right]w(S).$$

If $S\notin\mathcal{F}(N)$, the number of non-singleton supports of $\mathcal{F}_{S\cup B}$ is less than k, for all $B\in\mathcal{B}(\mathcal{F})$, $B\cap S\neq\emptyset$. By induction hypothesis, $\phi_j(S\cup B,v_{S\cup B},\mathcal{F}_{S\cup B})=\phi_j(S\cup B,v^{\mathcal{F}}_{S\cup B},\mathcal{F}_{S\cup B})$, and hence

$$\begin{aligned}\left(v^{\mathcal{F}}\right)^*(S)&=v^{\mathcal{F}}(S)+\sum_{j\in S^*\setminus S}\frac{\sum_{B\in B(j,S)}[\phi_j(S\cup B,v^{\mathcal{F}}_{S\cup B},\mathcal{F}_{S\cup B})-v^{\mathcal{F}}(\{j\})]}{|B(j,S)|}\\&=v^*(S).\end{aligned}$$

Moreover, $v^*(N)=(v^{\mathcal{F}})^*(N)$. Therefore,

$$w(S)=\begin{cases}0 & \text{if } S\notin\mathcal{F}(N)\text{ or } S=N,\\ \sum_{j\in N\setminus S}[\phi_j(N,v^{\mathcal{F}},\mathcal{F})-\phi_j(N,v,\mathcal{F})] & \text{otherwise.}\end{cases}$$

On the other hand, for each $S\in\mathcal{F}(N)$, the only coalitions $R\in\mathcal{F}$, $R\supseteq S$ are S and N if $i\in S$ and only N if $i\notin S$. Since $\mu(S,S)=1$ and $\mu(S,N)=-1$, we have that

$$\begin{aligned}&\sum_{\{R\in\mathcal{F}\colon R\supseteq S,i\in R\}}\mu(S,R)\phi_i(R,u_R,\mathcal{F}_R)\\&\quad=\begin{cases}\phi_i(S,u_S,\mathcal{F}_S)-\phi_i(N,u_N,\mathcal{F}) & \text{if } i\in S,\\ -\phi_i(N,u_N,\mathcal{F}) & \text{if } i\notin S.\end{cases}\end{aligned}$$

Therefore,

$$\begin{aligned}\phi_i(N,w,\mathcal{F})&=\sum_{S\in\mathcal{F}(i,N)}\phi_i(S,u_S,\mathcal{F}_S)w(S)-\sum_{S\in\mathcal{F}(N)}\phi_i(N,u_N,\mathcal{F})w(S)\\&=\sum_{S\in\mathcal{F}(i,N)}\phi_i(S,u_S,\mathcal{F}_S)\sum_{j\in N\setminus S}\left[\phi_j\left(N,v^{\mathcal{F}},\mathcal{F}\right)-\phi_j(N,v,\mathcal{F})\right]\\&\quad-\sum_{S\in\mathcal{F}(N)}\phi_i(N,u_N,\mathcal{F})\sum_{j\in N\setminus S}\left[\phi_j\left(N,v^{\mathcal{F}},\mathcal{F}\right)-\phi_j(N,v,\mathcal{F})\right]\end{aligned}$$

$$= \sum_{j\in N}\Bigg[\sum_{\substack{S\in\mathcal{F}(i,N)\\ j\in N\setminus S}} \phi_i(S,u_S,\mathcal{F}_S)\Bigg]\big[\phi_j\big(N,v^{\mathcal{F}},\mathcal{F}\big)-\phi_j(N,v,\mathcal{F})\big]$$
$$- \sum_{S\in\mathcal{F}(N)}\sum_{j\in N\setminus S}\big[\phi_j\big(N,v^{\mathcal{F}},\mathcal{F}\big)-\phi_j(N,v,\mathcal{F})\big]\phi_i(N,u_N,\mathcal{F}).$$

By A3 and A6, $\phi_j(N,v^{\mathcal{F}},\mathcal{F})-\phi_j(N,v,\mathcal{F})=\phi_j(N,w,\mathcal{F})$ and hence

$$\phi_i(N,w,\mathcal{F}) = \sum_{j\in N}\Bigg[\sum_{\substack{S\in\mathcal{F}(i,N)\\ j\in N\setminus S}} \phi_i(S,u_S,\mathcal{F}_S)\Bigg]\phi_j(N,w,\mathcal{F})$$
$$- \sum_{S\in\mathcal{F}(N)}\sum_{j\in N\setminus S}\phi_j(N,w,\mathcal{F})\phi_i(N,u_N,\mathcal{F}).$$

In matrix form,

$$\phi(N,w,\mathcal{F}) = M\phi(N,w,\mathcal{F}) - \lambda\phi(N,u_N,\mathcal{F}) \tag{4}$$

where $M=[m_{ij}]_{n\times n}$ is the matrix obtained in the proof of Theorem 8 and

$$\lambda = \sum_{S\in\mathcal{F}(N)}\sum_{j\in N\setminus S}\phi_j(N,w,\mathcal{F}).$$

We now distinguish two cases:

Case 1. There is only one non-singleton support, there is no articulation point and

$$\lambda = (n-1)\big[\phi_1(N,w,\mathcal{F})+\cdots+\phi_n(N,w,\mathcal{F})\big] = (n-1)w(N) = 0.$$

Hence, $(I-M)\phi(N,w,\mathcal{F})=\mathbf{0}$. As $|M-I|=(n-2)(-2)^{n-1}$, then $|M-I|\neq 0$ if $n>2$ and $\phi(N,w,\mathcal{F})=(I-M)^{-1}\mathbf{0}=\mathbf{0}$. If $n=2$, then $\phi_1(N,w,\mathcal{F})=\phi_2(N,w,\mathcal{F})=\alpha$. Therefore, $0=w(N)=\phi_1(N,w,\mathcal{F})+\phi_2(N,w,\mathcal{F})=2\alpha$ and $\phi(N,w,\mathcal{F})=\mathbf{0}$.

Case 2. There is more than one non-singleton supports, $\mathcal{F}$ contains at least one articulation point. The eigenvalue ν of M obtained by the application of the Perron–Frobenius theorem is strictly smaller than 1. Then, $M\phi(N,u_N,\mathcal{F})=\nu\phi(N,u_N,\mathcal{F})$ and hence $(I-M)\phi(N,u_N,\mathcal{F})=(1-\nu)\phi(N,u_N,\mathcal{F})$ where $(1-\nu)\phi(N,u_N,\mathcal{F})$ is strictly positive. Therefore M is a productive matrix (Bapat and Raghavan, 1997). Thus $(I-M)^{-1}$ exists and is a nonnegative matrix. Equation (4) can be written as

$$(I-M)\phi(N,w,\mathcal{F}) = -\lambda\phi(N,u_N,\mathcal{F})$$

and $\phi(N,w,\mathcal{F})=-\lambda(I-M)^{-1}\phi(N,u_N,\mathcal{F})$. We know that $\phi(N,u_N,\mathcal{F})>\mathbf{0}$ and the matrix $(I-M)^{-1}$ is nonnegative. Thus, the vector $(I-M)^{-1}\phi(N,u_N,\mathcal{F})$ is also strictly positive and the only possibility is $\lambda=0$ and $\phi(N,w,\mathcal{F})=\mathbf{0}$. In both cases, we have that

$$\mathbf{0} = \phi(N,w,\mathcal{F}) = \phi\big(N,\big(v^{\mathcal{F}}\big)^*,\mathcal{F}\big) - \phi(N,v^*,\mathcal{F})$$

and hence, by A6,

$$\phi\big(N,v^{\mathcal{F}},\mathcal{F}\big) = \phi\big(N,\big(v^{\mathcal{F}}\big)^*,\mathcal{F}\big) = \phi(N,v^*,\mathcal{F}) = \phi(N,v,\mathcal{F}). \qquad \square$$

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