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THE LOVÁSZ EXTENSION OF MARKET GAMES

ABSTRACT. The multilinear extension of a cooperative game was introduced by Owen in 1972. In this contribution we study the Lovász extension for cooperative games by using the marginal worth vectors and the dividends. First, we prove a formula for the marginal worth vectors with respect to compatible orderings. Next, we consider the direct market generated by a game. This model of utility function, proposed by Shapley and Shubik in 1969, is the concave biconjugate extension of the game. Then we obtain the following characterization: The utility function of a market game is the Lovász extension of the game if and only if the market game is supermodular. Finally, we present some preliminary problems about the relationship between cooperative games and combinatorial optimization.

KEY WORDS: Owen extension, Lovász extension, market games

1. INTRODUCTION

The Owen extension of a cooperative game is introduced by Owen (1972) as a multilinear polynomial in n real variables, where the coefficients are the dividends of the coalitions. The main applications of this extension were obtained with the construction of formulas for the Shapley and Banzhaf values. The Lovász extension of a convex game satisfies an optimization property on the core of the game. We show that this property implies a new formula for the Lovász extension of every cooperative game. Moreover, the coefficients of this extension are also the dividends of the coalitions. Market games and totally balanced games are the same ones and the utility function of the traders in a direct market is continuous and concave. In this paper we prove that the utility function is the Lovász extension of the game if and only if the game is supermodular.

Let us briefly outline the contents of our contribution. In the next section, we provide definitions and preliminaries results on

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cooperative game theory. In Section 3 we define market games and some of their classical properties are described. Section 4 is devoted to introduce the Lovász extension of a cooperative game and the link between duality for convex–concave functions and duality for submodular–supermodular functions. The precise statement was proved by Lovász (1983): A set function is submodular if and only if its Lovász extension is convex. Section 4 also includes the representation of Lovász extension using dividends. Section 5 offers the characterization of the utility function of the traders in a direct market. The final section contains an outline of the implications and consequences for game theory of this new link between cooperative game theory and combinatorial optimization.

2. PRELIMINARIES

A cooperative game (N, v) is a function $v : 2^N \to R$ with $v(\emptyset) = 0$. The players are the elements of the finite set $N = \{1, \ldots, n\}$, and the coalitions are the subsets $S \subseteq N$. To every coalition S is associated its characteristic vector e^S, defined by

$$(e_i^S)_{i \in N} = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

This is the natural correspondence between 2^N and $\{0, 1\}^n$. Through this identification of coalitions with their characteristic vectors, a cooperative game is a function $v : \{0, 1\}^n \to R$ with $v(\mathbf{0}) = 0$.

Unanimity games are considered in order to obtain several extensions of a cooperative game in terms of its dividends. For any $S \subseteq N$, $S \neq \emptyset$,

$$u_S(T) = \begin{cases} 1 & \text{if } T \supseteq S, \\ 0 & \text{otherwise} \end{cases}$$

is called the S-unanimity game. Every game v is a linear combination of unanimity games, that is,

$$v = \sum_{S \subseteq N} a_S(v) u_S,$$

where the coefficients $\{a_S(v) : S \subseteq N, S \neq \emptyset\}$ are the *dividends* of the coalition S in the game v. Any cooperative game (N, v) has a unique expression as a multilinear polynomial in n discrete variables:

$$v(x) = \sum_{S \subseteq N} \left(a_S(v) \prod_{i \in S} x_i \right), \quad x \in \{0, 1\}^n.$$

This polynomial expression using real variables was introduced in game theory by Owen (1972) as the *multilinear extension* $f : [0, 1]^n \to R$, of the cooperative game $v : \{0, 1\}^n \to R$. Then, we have $f(e^S) = v(S)$ for all $S \subseteq N$. Owen showed that the Shapley and Banzhaf values are given by

$$\Phi_i(N, v) = \int_0^1 \frac{\partial f}{\partial x_i}(t, \ldots, t) \, \mathrm{d}t,$$

$$\beta_i(N, v) = \frac{\partial f}{\partial x_i}\left(\frac{1}{2}, \ldots, \frac{1}{2}\right), \quad 1 \leq i \leq n.$$

Let us recall some classical solution concepts for cooperative games. The *core* of a game (N, v) is the set

$$\mathrm{Core}(v) = \{x \in R^n : x(N) = v(N), x(S) \geq v(S) \text{ for all } S \subset N\},$$

where for any $S \subseteq N, x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$.

Let us assume a total ordering of the elements of N, defined by $i_1 < i_2 < \cdots < i_n$. Given the previous ordering C, consider the following chain of coalitions:

$$C_0 \subset C_1 \subset \cdots \subset C_{n-1} \subset C_n,$$

where $C_0 = \emptyset$ and $C_k = \{i_1, i_2, \ldots, i_k\}$, $k = 1, \ldots, n$. The *marginal worth vector* $a^C \in R^n$ with respect to the ordering C in the game (N, v) is given by

$$a^C_{i_k} = v(C_k) - v(C_{k-1}), \quad k = 1, \ldots, n.$$

The *Weber set* of the game (N, v) is the convex hull of the marginal worth vectors, i.e., $\mathrm{Weber}(v) = \mathrm{conv}\{a^C : C$ is an *ordering of* $N\}$. Convex games were introduced by Shapley (1971). A game (N, v) is convex if for every $S, T \in 2^N$,

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

It is easy to prove that $a^C(C_k) = v(C_k)$ for $k = 1, \ldots, n$. Weber (1988) showed that any game satisfies $\text{Core}(v) \subseteq \text{Weber}(v)$ and Ichiishi (1981) proved that if $\text{Weber}(v) \subseteq \text{Core}(v)$ then (N, v) is a convex game. Thus, these results imply the following characterization of convex games.

THEOREM 1. *The cooperative game* (N, v) *is convex if and only if* $\text{Core}(v) = \text{Weber}(v)$.

3. MARKET GAMES

We consider a game model of a pure exchange economy. Let $N = \{1, \ldots, n\}$ be the set of traders and we suppose that they participate in a market encompassing trade in m commodities. The space R^m_+ is considered as the commodity space. Every trader $i \in N$ is characterized by means of an initial endowment vector $w^{(i)} \in R^m_+$ and by a utility function $u_i : R^m_+ \to R$ which measures the worth, for him, of any bundle of commodities. The individual utility functions u_i are continuous and concave. The four components vector $\mathbf{M} = (N, m, \{w^{(i)}\}_{i \in N}, \{u_i\}_{i \in N})$ is called a *market*.

We denote by $w(S) = \sum_{i \in S} w^{(i)} \in R^m_+$ the aggregate endowment of the coalition of traders S. Then, $w(S)$ can be reallocated as a collection $\{a^{(i)} : i \in S\}$ of bundles such that each $a^{(i)} \in R^m_+$ and $a(S) = \sum_{i \in S} a^{(i)} = w(S)$. Denote by $A(S)$ the set of these collections. Since the individual utility functions are continuous and $A(S)$ is a compact set, we can define a cooperative game $(N, v_{\mathbf{M}})$ as

$$v_{\mathbf{M}}(S) = \max\left\{ \sum_{i \in S} u_i(a^{(i)}) : \{a^{(i)} : i \in S\} \in A(S) \right\},$$

for all $S \subseteq N$. This model is called a *market game* (see Kannai, 1992), and it corresponds to the original market in a natural way.

A game (N, v) is *totally balanced* if for all $S \subseteq N$ and all e^S-balanced collection, i.e., $\{\gamma_T\}_{T \subseteq N}$ with $\sum_{T \subseteq N} \gamma_T e^T = e^S$ and $\gamma_T \geq 0$, satisfies $\sum_{T \subseteq N} \gamma_T v(T) \leq v(S)$. The following result is due to Shapley and Shubik (1969).

THEOREM 2. *A game is a market game if and only if it is totally balanced.*

4. THE LOVÁSZ EXTENSION

We now consider an extension of a cooperative game (N, v) to a function on R^n. We say that a non-negative function $f: 2^N \to R_+$ is a weighted chain if the family $\Omega = \{S \subseteq N : f(S) > 0\}$ is a chain, i.e., $S \subseteq T$ or $T \subseteq S$ for every pair $S, T \in \Omega$. To every weighted chain f, we associate a non-negative vector

$$x = \sum_{S \subseteq N} f(S) e^S \in R^n_+$$

called the depth vector of f. This is a one-to-one correspondence: for a non-negative vector $x \in R^n_+$ let $0 \leq x^1 < x^2 < \cdots < x^k$ be the components of x with different value and let $S_p = \{i \in N : x_i \geq x^p\}$. Then we define

$$f_x(S) = \begin{cases} x^p - x^{p-1} & \text{if } S = S_p, \\ 0 & \text{otherwise}, \end{cases}$$

where $x^0 = 0$. Obviously f_x is a weighted chain and its depth vector is x.

Let (N, v) be a cooperative game. There is a natural way of extending $v : 2^N \to R$ to all non-negative real vectors.

DEFINITION 1. Let $x \in R^n_+$ be a non-negative vector and f_x its weighted chain, the Lovász extension of v is $\hat{v} : R^n_+ \to R$ defined by $\hat{v}(x) = \sum_{S \subseteq N} f_x(S) v(S)$.

The function $\hat{v}$ is an extension of v because $\hat{v}(e^S) = v(S)$ for all $S \subseteq N$ and it has the following properties:

1. $\hat{v}$ is positively homogeneous, i.e., $\hat{v}(\lambda x) = \lambda \hat{v}(x)$ for all $\lambda \geq 0$.
2. $\widehat{v_1 + v_2} = \widehat{v_1} + \widehat{v_2}$.
3. $\widehat{\lambda v} = \lambda \hat{v}$ for all $\lambda \in R$.

Lovász (1983) proved the following characterization of the supermodular functions.

THEOREM 3. *A function $v : 2^N \to R$ is supermodular if and only if the Lovász extension $\hat{v}$ of v is concave.*

Note that convex games are supermodular functions with $v(\emptyset) = 0$. Moreover, the Lovász extension of a supermodular function satisfies (see Fujishige, 1991) the next optimization property: $\hat{v}(x) = \min\{\langle x, y\rangle : y \in P(v)\}$ for all $x \in R^n_+$, where

$$P(v) = \{y \in R^n : y(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

The Lovász extension is strongly related to the greedy algorithm. An ordering $i_1 < i_2 < \cdots < i_n$ is compatible with the vector $x \in R^n_+$ if $x_{i_1} \geq x_{i_2} \geq \cdots x_{i_n} \geq 0$. We obtain a formula for the marginal worth vectors with respect to x-compatible orderings in convex games.

THEOREM 4. *Let (N, v) be a convex game and let $x \in R^n_+$. The marginal worth vector a^C with respect to an x-compatible ordering C satisfies*

$$\hat{v}(x) = \min\{\langle x, y\rangle : y \in \mathrm{Core}(v)\} = \langle x, a^C\rangle.$$

Proof. If v is a convex game then Theorem 1 implies that $a^C \in Core(v) \subset P(v)$ and hence we obtain the result if we prove $\langle x, y\rangle \geq \langle x, a^C\rangle$ for all $y \in P(v)$. If $y \in P(v)$ then the summation by parts implies that

$$\begin{aligned}
\langle x, y\rangle &= \sum_{k=1}^{n} x_{i_k} y_{i_k} = \sum_{k=1}^{n-1} \left[(x_{i_k} - x_{i_{k+1}}) \sum_{j=1}^{k} y_{i_j} \right] + x_{i_n} \sum_{j=1}^{n} y_{i_j} \\
&\geq \sum_{k=1}^{n-1} (x_{i_k} - x_{i_{k+1}}) v(C_k) + x_{i_n} v(C_n) \\
&= \sum_{k=1}^{n} x_{i_k} [v(C_k) - v(C_{k-1})] \\
&= \sum_{k=1}^{n} x_{i_k} a^C_{i_k} = \langle x, a^C\rangle. \qquad \square
\end{aligned}$$

Our next theorem gives the formula for computing the Lovász extension using dividends.

THEOREM 5. *Let* (N, v) *be a game with dividends* $\{a_S(v) : S \subseteq N,\ S \neq \emptyset\}$ *. Then the Lovász extension of v satisfies* $\hat{v}(x) = \sum_{S \subseteq N} a_S(v) \min_{i \in S} x_i$, *for all* $x \in R^n_+$.

Proof. Properties 2 and 3 of the Lovász extension imply that $\hat{v} = \sum_{S \subseteq N} a_S(v) \widehat{u_S}$. Every unanimity game u_S is a convex game and we can use the optimization property showed in Theorem 4. Thus,

$$\begin{aligned} \widehat{u_S}(x) &= \min\{\langle x, y\rangle : y \in \mathrm{Core}(u_S)\} \\ &= \min\{\langle x, e_i\rangle : i \in S\} = \min_{i \in S} x_i, \end{aligned}$$

where the second equation follows from the characterization of the core for unanimity games (see Einy and Wettstein, 1996):

$$\mathrm{Core}(u_S) = \mathrm{Weber}(u_S) = \mathrm{conv}\{e_i : i \in S\}. \qquad \square$$

REMARK. Driessen and Rafels (1999) have studied several characterizations for the Lovász extension of k-convex games.

5. THE UTILITY FUNCTION

For every totally balanced game (N, v) we consider the following direct market $\mathbf{M}_0 = (N, n, \{e^i\}_{i \in N}, u)$, where u is the same utility function for all traders, given by

$$u(x) = \max\left\{ \sum_{T \subseteq N} \gamma_T v(T) : \{\gamma_T\}_{T \subseteq N} \text{ is an } x\text{-balanced collection} \right\}.$$

This utility funcion is the concave biconjugate extension $v^{\circ\circ}$ of the game v (see Fujishige, 1991). Every convex game is totally balanced. In this case we obtain the following property of the Lovász extension.

THEOREM 6. *Let* (N, v) *be a totally balanced game with common utility function* $u : R^n_+ \to R$. *Then* (N, v) *is convex if and only if its Lovász extension* $\hat{v} = u$.

Proof. The utility function is the solution of the next linear programming problem: $u(x) = \max\{\langle \gamma, v\rangle : A\gamma = x,\ \gamma \geq 0\}$, where $\gamma = (\gamma_T)_{T \subseteq N}$, $v = (v(T))_{T \subseteq N}$ and the matrix $A = (e_i^T)_{i \in N, T \subseteq N}$. The dual problem is

$$u(x) = \min\{\langle x, y \rangle : y^T A^T \geq v\} = \min\{\langle x, y \rangle : y \in P(v)\}.$$

If the game (N, v) is convex then v is supermodular and the optimization property implies $\hat{v} = u$. Conversely, if the utility function satisfies $u = \hat{v}$ then $\hat{v}$ is concave and Theorem 3 implies that (N, v) is a convex game. □

6. OPEN PROBLEMS

The optimization of non-linear functions over the core of a submodular or supermodular function was considered by Fujishige (1991) who extended the Fenchel's duality theory. Also, he introduced the concept of subdifferential of a submodular function. A theory for discrete optimization (non-linear integer programming), called *discrete convex analysis* was developed by Murota (1998). This theory includes functions defined on the integral points in the core of a game.

Some problems about the new relationship between cooperative games and combinatorial optimization are:

1. To apply the Fenchel min–max duality theory and Murota's discrete convex analysis to game theory.
2. To introduce the subgradient vectors and subdifferential sets (see Fujishige, 1984) in cooperative game theory.
3. To study the relationships between the Shapley and Banzhaf values of a game and the subdifferential of its Lovász extension.

Martínez-Legaz (1996) introduced the *indirect function* $\pi : R^n \to R$ of a cooperative game (N, v) by $\pi(x) = \max\{v(S) - x(S) : S \subseteq N\}$, for all $x \in R^n$. Note that the utility function u of the direct market induced by a totally balanced game satisfies

$$u(x) = \min\{\langle x, y \rangle : y \in P(v)\} = \min\{\langle x, y \rangle : y \in \pi^{-1}(0)\}.$$

We propose the following two additional questions for market games:

4. Is there another extension with the utility function property for totally balanced games?
5. To study the relationship between the indirect function of a game and its Lovász extension.

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