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The position value in communication structures

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Abstract. We study cooperation structures with the following property: given any two feasible coalitions with non-empty intersection, its union is a feasible coalition again. TU-games restricted by union stable systems generalize graph-restricted games and games with permission structures. A study about the differences between the position value in union stable systems and hypergraph communication situations is given. Moreover, some computational aspects related to position value in union stable systems are discussed.

Key words: Position value, Communication situations, Permission structures, Hypergraph communication situations, Union stable systems

Mathematics Subject Classification (2000): 91A12

1 Introduction

In cooperative game theory, partial cooperation assumes that the formation of any player coalition is not possible. From this principle, several models have been proposed, among them we are interested in those models derived from the *communication situations* introduced by Myerson [7]. This line of research was continued by Owen [11], Borm, Owen and Tijs [3], van den Nouweland, Borm and Tijs [9] and Potters and Reijnierse [12].

In the Myerson model, the bilateral relations among the players are represented by means of an undirected graph and the feasible coalitions are those that induce connected subgraphs. However, partial cooperation can not always be modelled by a graph, (it would be the case if there were no feasible

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coalitions with less than three players). As a consequence, some ideas of the communication model initiated by Myerson have been generalized in several directions. For instance, *conference structures* by Myerson [8], *hypergraph communication situations* by van den Nouweland, Borm and Tijs [9] and *union stable systems* by Algaba, Bilbao, Borm and López [1, 2].

In Algaba et al. [1, 2], it is assumed that if two feasible coalitions have common elements, these ones will act as intermediaries between the two coalitions in order to establish meaningful cooperation in the union of these coalitions. These feasible coalition systems are called *union stable systems*. This mathematical feature will be essential in our study and it is satisfied for the feasible coalitions coming from graph communication situations and *permission structures* (see [4, 5]). Furthermore, these systems have a close relation with the hypergraph communication situations.

Section 2 recalls the main definitions on restricted cooperation by means of union stable systems including the crucial driving notion of *basis* and *position value* as well as the description of hypergraph communication situations and its position value. In Section 3, we study the relationship between hypergraph communication situations and union stable systems, with the objective to point out the necessity of the basis. This justification is based on that given a hypergraph communication situation, a union stable system is associated, although the same union stable system can be arisen by different hypergraphs. This reason has an essential importance to determine the position value, since this one is an allocation rule neither *fair* nor *stable*. Section 4 deals with some computational aspects about the position value in union stable systems. So, a method for the *dividend* calculation of the *conference game* is given. On the other hand, and taking into account that the coalition N may not be feasible, we prove that the calculation of the position value can be simplified and computed by the maximal feasible coalition of N which each player belongs to.

2 Communication structures

In this section, we recall the main definitions and concepts related to union stable systems and hypergraph communication situations whose relation will be analyzed in the next section.

Let $N = \{1, \ldots, n\}$ be a finite set of players and $\mathcal{F} \subseteq 2^N$ a system of feasible coalitions. The set system $\mathcal{F}$ is called *union stable* if for all $A, B \in \mathcal{F}$ with $A \cap B \neq \emptyset$ it is satisfied that $A \cup B \in \mathcal{F}$.

A *communication situation* is a triple (N, v, E), where (N, v) is a game and (N, E) is a simple graph. It is easy to see that the set system $\mathcal{F}$, defined by

$$\mathcal{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } (N, E)\},$$

is union stable. However, a union stable system can not always be modelled by a communication situation.

Let $\mathcal{F}$ be a union stable system and $\mathcal{G} \subseteq \mathcal{F}$. We define inductively the families

$$\mathcal{G}^{(0)} = \mathcal{G}, \quad \mathcal{G}^{(n)} = \left\{ S \cup T : S, T \in \mathcal{G}^{(n-1)},\ S \cap T \neq \emptyset \right\}, \quad (n = 1, 2, \ldots).$$

Notice that $\mathcal{G}^{(0)} \subseteq \mathcal{G}^{(n-1)} \subseteq \mathcal{G}^{(n)} \subseteq \mathcal{F}$, since $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{F}$ is union stable. We define $\overline{\mathcal{G}}$ by $\overline{\mathcal{G}} = \mathcal{G}^{(k)}$, where k is the smallest integer such that $\mathcal{G}^{(k+1)} = \mathcal{G}^{(k)}$.

For each union stable family $\mathcal{F}$, we are interested in finding a minimal subset $\mathcal{B}(\mathcal{F})$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$. So, we can consider the following set:

$$\mathcal{D}(\mathcal{F}) = \{G \in \mathcal{F} : G = A \cup B,\ A \neq G,\ B \neq G,\ A, B \in \mathcal{F},\ A \cap B \neq \emptyset\}.$$

The set $\mathcal{B}(\mathcal{F}) = \mathcal{F} \setminus \mathcal{D}(\mathcal{F})$ is called the *basis* of $\mathcal{F}$ and the elements of $\mathcal{B}(\mathcal{F})$ are called supports of $\mathcal{F}$.

We remark that the basis $B(\mathcal{F})$ is the minimal subset of the union stable system $\mathcal{F}$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$ (see Algaba et al. [1]).

Let $\mathcal{G} \subseteq 2^N$ be a set system and let $S \subseteq N$. A set $T \subseteq S$ is called a $\mathcal{G}$-*component* of S if it is satisfied that $T \in \mathcal{G}$ and there exists no $T' \in \mathcal{G}$ such that $T \subset T' \subseteq S$.

Therefore, the $\mathcal{G}$-components of S are the maximal feasible coalitions that belong to $\mathcal{G}$ and are contained in S. We denote by $C_{\mathcal{G}}(S)$ the collection of the $\mathcal{G}$-components of S . Union stable systems can be characterized in terms of the $\mathcal{F}$-components of a coalition in the following way:

The set system $\mathcal{F} \subseteq 2^N$ is union stable if and only if for any $S \subseteq N$ with $C_{\mathcal{F}}(S) \neq \emptyset$, the $\mathcal{F}$-components of S are a collection of pairwise disjoint subsets of S.

Let (N, v) be a cooperative game and $\mathcal{F} \subseteq 2^N$ a union stable system. Let $\mathcal{B}$ be the basis of $\mathcal{F}$ and $\mathcal{C} = \{B \in \mathcal{B} : |B| \geq 2\}$. The $\mathcal{F}$- *restricted game* $v^{\mathcal{F}} : 2^N \to \mathbb{R}$, is defined by $v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$ and the *conference game* is the game $(\mathcal{C}, v^{\mathcal{C}})$ where $v^{\mathcal{C}} : 2^{\mathcal{C}} \to \mathbb{R}$, is defined by $v^{\mathcal{C}}(\mathcal{A}) = v^{\overline{\mathcal{A}}}(N)$.

Note that the game $(\mathcal{C}, v^{\mathcal{C}})$ is well defined since for each $\mathcal{A} \subseteq \mathcal{C}$, $\overline{\mathcal{A}}$ is a union stable system. The $\mathcal{F}$-restricted game focuses on the role of a player in creating economic possibilities and establishing meaningful communication among the players whereas the conference game measures the economic value of the grand coalition when specific parts of the cooperation structure are considered.

The two above definitions extend the *point game* and the *arc game* respectively. The arc game was introduced by Borm, Owen, and Tijs [3] and for a communication situation (N, v, E) we have that $\mathcal{C} = \{\{i, j\} : \{i, j\} \in E\}$.

A *union stable cooperation structure* is a triple $(N, v, \mathcal{F})$ where $N = \{1, \ldots, n\}$ is the set of players, (N, v) is a game $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, and $\mathcal{F}$ a union stable system.

The position value for graph communication situations was first introduced in Meesen [6] and studied in Borm, Owen and Tijs [3]. Next, we recall the definition of this value for union stable systems (see [1]). Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. For $i \in N$ the *position value* $\pi_i(N, v, \mathcal{F})$ is given by

$$\pi_i(N, v, \mathcal{F}) = \sum_{C \in \mathcal{C}_i} \frac{1}{|C|} \Phi_C(\mathcal{C}, v^{\mathcal{C}}),$$

where $\mathcal{C}_i = \{C \in \mathcal{C} : i \in C\}$.

As we have indicated, communication situations (N, v, E) are a particular case of union stable cooperation structures $(N, v, \mathcal{F})$. However, these structures are not the unique line of generalization of communication situations.

So, Myerson [8] generalized his idea of modelling the partial cooperation by a communication graph between player pairs, by means of the so-called *conference structures*, although he considered *NTU*-games. On the other hand, van den Nouweland, Borm and Tijs [9] extended the Myerson's original idea towards *hypergraph communication situations*.

In [2] was established a relation between conference structures à la Myerson and union stable systems. To illustrate the relation between union stable cooperation structures and hypergraph communication situations we first recall the general notions about hypergraph communication situations. *A hypergraph communication situation* is a triple $(N, v, \mathcal{H})$ where (N, v) is a zero-normalized TU-game and $(N, \mathcal{H})$ is a hypergraph, with $\mathcal{H} \subseteq \{H \in 2^N : |H| \geq 2\}$. In these structures the communication is only possible through the *conferences* $H \in \mathcal{H}$. So, given a coalition $S \subseteq N$, the *interaction sets* or feasible coalitions of the coalition S are defined in the following way:

1. For all $i \in S$, $\{i\}$ is an interaction set of S.
2. If $H \in \mathcal{H}$ and $H \subseteq S$, then H is an interaction set of S.
3. If T_1 and T_2 are interaction sets of S and satisfying that $T_1 \cap T_2 \neq \emptyset$, then $T_1 \cup T_2$ is an interaction set of S.

Let $S \subseteq N$ and consider the partially ordered set formed by the interaction sets of S with the inclusion relation. By the formation process of interaction sets of S is deduced that its maximal elements give rise to a partition of S denoted by $S/\mathcal{H}$ and then the *restricted game by the communication hypergraph*, $\left(N, r^v_{\mathcal{H}}\right)$ is given by

$$r^v_{\mathcal{H}} : 2^N \to \mathbb{R}, \quad r^v_{\mathcal{H}}(S) = \sum_{C \in S/\mathcal{H}} v(C), \ \forall S \subseteq N.$$

The *conference game*, $\left(\mathcal{H}, r^v_N\right)$, is determined by

$$r^v_N : 2^{\mathcal{H}} \to \mathbb{R}, \quad r^v_N(\mathcal{A}) = r^v_{\mathcal{A}}(N) = \sum_{C \in N/\mathcal{A}} v(C), \forall \mathcal{A} \subseteq \mathcal{H}.$$

The value of any conference subset, $\mathcal{A}$, is the value of coalition N in the restricted game corresponding to the hypergraph communication situation $(N, v, \mathcal{A})$.

If $(N, v, \mathcal{H})$ is a hypergraph communication situation, the position value $\pi(N, v, \mathcal{H})$ is defined [9], for $i \in N$, as

$$\pi_i(N, v, \mathcal{H}) = \sum_{H \in \mathcal{H}_i} \frac{1}{|H|} \Phi_H\left(\mathcal{H}, r^v_N\right),$$

where $\mathcal{H}_i = \{H \in \mathcal{H} : i \in H\}$.

For convenience, we assume from now on that the underlying game (N, v) is zero-normalized.

3 The position value in union stable systems and hypergraph communication situations

The aim of this section is to establish the relation between hypergraph communication situations and union stable cooperation structures, emphasizing on the position value defined on these communication structures. The next theorem establishes this relation and later we provide two examples to compare the differences between the position value in union stable systems and hypergraph communication situations. In the first of them we want to underline that the family of feasible coalitions can be modelled by a graph and the position value corresponding to the union stable cooperation structure and the graph communication situation one coincide and these are different from the one defined on hypergraph communication situations. The second example makes clear that it is not the same to consider the position value on any hypergraph than on the hypergraph formed by the non-unitary supports. This second example can not be modelled by a communication situation.

Theorem 3.1 *Let $(N, v, \mathcal{H})$ be a hypergraph communication situation. There exist a union stable cooperation structure $(N, v, \mathcal{F}(\mathcal{H}))$ such that the corresponding restricted games $\left(N, v^{\mathcal{F}(\mathcal{H})}\right)$ and $\left(N, r^v_{\mathcal{H}}\right)$ coincide, and the conference games $\left(\mathcal{C}, v^{\mathcal{C}}\right)$ and $\left(\mathcal{H}, r^v_N\right)$ satisfy*

$$\forall \mathcal{A} \subseteq \mathcal{H}, \quad \exists \mathcal{C}' \subseteq \mathcal{C} \quad \text{such that} \quad v^{\mathcal{C}}(\mathcal{C}') = r^v_N(\mathcal{A}).$$

Conversely, if $(N, v, \mathcal{F})$ is a union stable cooperation structure, then $(N, v, \mathcal{C})$ is a hypergraph communication situation where it holds that $r^v_{\mathcal{C}} = v^{\mathcal{F}}$ and $r^v_N = v^{\mathcal{C}}$.

Proof. Let $(N, v, \mathcal{H})$ be a hypergraph communication situation and consider the set $\mathcal{F}(\mathcal{H})$ formed by all interaction sets of coalition N. By construction, $\mathcal{F}(\mathcal{H})$ is a union stable system and, in the cooperation structure $(N, v, \mathcal{F}(\mathcal{H}))$, the definition of $\mathcal{F}(\mathcal{H})$-restricted game $\left(N, v^{\mathcal{F}(\mathcal{H})}\right)$ coincides with the corresponding to the restricted game by the communication hypergraph $\left(N, r^v_{\mathcal{H}}\right)$ since, for all coalition $S \subseteq N$, $C_{\mathcal{F}(\mathcal{H})}(S) = S/\mathcal{H}$.

Consider now $\mathcal{A} \in 2^{\mathcal{H}}$. By definition, $r^v_N(\mathcal{A}) = r^v_{\mathcal{A}}(N) = \sum_{M \in N/\mathcal{A}} v(M)$. In the union stable system, $\mathcal{F}(\mathcal{H})$, let $\mathcal{F}' = \{F \in \mathcal{F}(\mathcal{H}) : F \subseteq M$ for some $M \in N/\mathcal{A}\}$.

The system $\mathcal{F}'$ is union stable and its basis $\mathcal{B}'$ is contained in the basis $\mathcal{B}$ of the union stable system $\mathcal{F}(\mathcal{H})$. Indeed, otherwise there would be an element $B \in \mathcal{B}'$ that does not belong to the basis $\mathcal{B}$, which means $\exists S, T \in \mathcal{F}(\mathcal{H})$, $S, T \neq B$ with $S \cap T \neq \emptyset$ and $S \cup T = B$. As $B \in \mathcal{B}' \subseteq \mathcal{F}'$ then $S, T \in \mathcal{F}'$ and therefore, $B \notin \mathcal{B}'$ which is a contradiction.

It is satisfied that $v^{\mathcal{C}}(\mathcal{C}') = r^v_N(\mathcal{A})$, where $\mathcal{C}$ and $\mathcal{C}'$ are formed by those supports with at least two elements in $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}'$ respectively.

Obviously $\mathcal{C}' \subseteq \mathcal{C}$ and we have that $\mathcal{B}' = \mathcal{F}'$ and $C_{\mathcal{F}'}(N) = \{M : M \in N/\mathcal{A}\}$. Moreover, $v^{\mathcal{C}}(\mathcal{C}') = v^{\mathcal{F}'}(N) = r^v_N(\mathcal{A})$. The proof of the converse is straightforward. □

It is immediate from the given definitions concerning the hypergraph communication situations and the above result that these ones lead to union stable cooperation structures, where the feasible coalitions are the so-called interaction sets of coalition N. However, given a union stable cooperation structure that contains the unitary coalitions, there exist several hypergraph communication situations, whose interaction sets corresponding to coalition N coincide with the feasible coalitions of the union stable system considered, (obviously, one of the hypergraph communication situations will be the one where the conference set $\mathcal{H}$ is formed by the non-unitary supports of the basis).

Next, we give an example to compare the position value defined on the different communication structures considered. Note that the family of feasible coalitions can be modelled by a graph.

Example 3.1 Let $N = \{1,2,3,4,5\}$ and $v : 2^N \to \mathbb{R}$ such that $v(S) = |S| - 1$, for all non-empty coalition S. Consider the hypergraph

$$\mathcal{H} = \{\{1,2\},\{2,3\},\{1,2,3\},\{4,5\}\},$$

where the interaction sets for coalition N are

$$\{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{2,3\},\{1,2,3\},\{4,5\}\}.$$

The Shapley value associated to the conference game is determined by

$$\Phi_{\{1,2\}}(\mathcal{H}, r_N^v) = \Phi_{\{2,3\}}(\mathcal{H}, r_N^v) = \frac{1}{2}, \quad \Phi_{\{1,2,3\}}(\mathcal{H}, r_N^v) = \Phi_{\{4,5\}}(\mathcal{H}, r_N^v) = 1,$$

and, therefore, the position value is $\pi(N, v, \mathcal{H}) = \left(\frac{7}{12}, \frac{10}{12}, \frac{7}{12}, \frac{1}{2}, \frac{1}{2}\right)$.

On the other hand, the set of feasible coalitions of the union stable cooperation structure $(N, v, \mathcal{F}(\mathcal{H}))$ coincides with the interaction sets for coalition N and, hence,

$$\mathcal{B}=\{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{2,3\},\{4,5\}\} \text{and } \mathcal{C}=\{\{1,2\},\{2,3\},\{4,5\}\}.$$

The Shapley value, for each non-unitary support is

$$\Phi_{\{1,2\}}(\mathcal{C}, v^{\mathcal{C}}) = \Phi_{\{2,3\}}(\mathcal{C}, v^{\mathcal{C}}) = \Phi_{\{4,5\}}(\mathcal{C}, v^{\mathcal{C}}) = 1$$

and, as a consequence, the position value is

$$\pi(N, v, \mathcal{F}) = \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),$$

which coincides with the one corresponding to the communication situation given.

Suppose the hypergraph communication situations $(N, v, \mathcal{H})$ and $(N, v, \mathcal{H}')$, where there exists $C \in \mathcal{H}$ such that $C = A \cup B$, with $A, B \in \mathcal{H}$, $A, B \neq C$, $A \cap B \neq \emptyset$ and $\mathcal{H}' = \mathcal{H} \setminus \{C\}$. The next example illustrates the interest in changing from $(N, v, \mathcal{H})$ to $(N, v, \mathcal{H}')$ since the players from the intersection will not accept $(N, v, \mathcal{H})$ instead of $(N, v, \mathcal{H}')$ because the profits in the maximal feasible coalitions of the grand coalition are the same and in $(N, v, \mathcal{H}')$ players have a better position. Therefore, it is enough that the elements in the intersection move away from the conference $C = A \cup B$.

Example 3.2 Consider the player set $N = \{1,2,3,4,5,6,7,8,9\}$ and the hypergraph communication situations $(N, v, \mathcal{H})$ and $(N, v, \mathcal{H}')$, where

$$\mathcal{H} = \{\{1,2,3,4\},\{3,4,7\},\{3,4,5,6\},\{1,2,3,4,5,6\},\{8,9\}\},$$

$$\mathcal{H}' = \{\{1,2,3,4\},\{3,4,7\},\{3,4,5,6\},\{8,9\}\} = \mathcal{H} \setminus \{1,2,3,4,5,6\}, \text{ and}$$

$$v : 2^N \to \mathbb{R}, \quad v(S) = \begin{cases} |S|^2 & \text{if } |S| \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

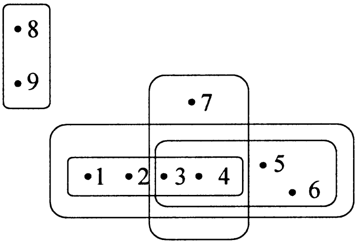

Figure 1

	$(N, v, \mathcal{H})$	$(N, v, \mathcal{H}')$	Δ	
$\Phi_{\{1,2,3,4\}}$	9	58/3	+31/3	
$\Phi_{\{3,4,5,6\}}$	9	58/3	+31/3	
$\Phi_{\{3,4,7\}}$	34/3	31/3	−1	$\sum \Delta = 59/3$
$\Phi_{\{1,2,3,4,5,6\}}$	59/3	–	–	
$\Phi_{\{8,9\}}$	4	4	0	

	$(N, v, \mathcal{H})$	$(N, v, \mathcal{H}')$	Δ
π_1	199/36	174/36	−25/36
π_2	199/36	174/36	−25/36
π_3	416/36	472/36	56/36
π_4	416/36	472/36	56/36
π_5	199/36	174/36	−25/36
π_6	199/36	174/36	−25/36
π_7	136/36	124/36	−12/36
π_8	2	2	0
π_9	2	2	0

In both cases, the feasible coalitions or interaction sets of N are given by

$$\begin{aligned}\mathcal{F} =&\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{1,2,3,4\},\{3,4,7\},\\&\{3,4,5,6\},\{1,2,3,4,5,6\},\{8,9\},\{1,2,3,4,7\},\{3,4,5,6,7\},\\&\{1,2,3,4,5,6,7\}\}.\end{aligned}$$

The conference games $\left(\mathcal{H}, v^{\mathcal{H}}\right)$ and $\left(\mathcal{H}', v^{\mathcal{H}'}\right)$ are tabulated on the next page. Notice that there are two maximal feasible coalitions of N: $\{1,2,3,4,5,6,7\}$ and $\{8,9\}$.

4 Computational aspects

One of the main problems of the position value is its computation. In this section, first and since dividends are decisive in the determination of the Shapley value, we make a study of them and some of the conditions to obtain easier expressions. This implies the computation of the dividends of the conference game $\left(\mathcal{C}, v^{\mathcal{C}}\right)$. On the other hand, and taking into account that the coalition N may not be feasible, we prove that the calculation of the position value can be simplified and computed by the restriction to the maximal

$\mathcal{A} \subseteq \mathcal{H}$	$\mathcal{A} \subseteq \mathcal{H}'$	$C_{\mathcal{F}}(N)^*$	$v^{\mathcal{H}}(\mathcal{A})$
$\emptyset$		$\emptyset$	0
$\{1,2,3,4\}$		$\{1,2,3,4\}$	16
$\{3,4,7\}$		$\{3,4,7\}$	9
$\{3,4,5,6\}$		$\{3,4,5,6\}$	16
$\{1,2,3,4,5,6\}$	—	$\{1,2,3,4,5,6\}$	36
$\{8,9\}$		$\{8,9\}$	4
$\{1,2,3,4\},\{3,4,7\}$		$\{1,2,3,4,7\}$	25
$\{1,2,3,4\},\{3,4,5,6\}$		$\{1,2,3,4,5,6\}$	36
$\{1,2,3,4\},\{1,2,3,4,5,6\}$	—	$\{1,2,3,4,5,6\}$	36
$\{1,2,3,4\},\{8,9\}$		$\{1,2,3,4\},\{8,9\}$	20
$\{3,4,7\},\{3,4,5,6\}$		$\{3,4,5,6,7\}$	25
$\{3,4,7\},\{1,2,3,4,5,6\}$	—	$\{1,2,3,4,5,6,7\}$	49
$\{3,4,7\},\{8,9\}$		$\{3,4,7\},\{8,9\}$	13
$\{3,4,5,6\},\{1,2,3,4,5,6\}$	—	$\{1,2,3,4,5,6\}$	36
$\{3,4,5,6\},\{8,9\}$		$\{3,4,5,6\},\{8,9\}$	20
$\{1,2,3,4,5,6\},\{8,9\}$	—	$\{1,2,3,4,5,6\},\{8,9\}$	40
$\{1,2,3,4\},\{3,4,7\},\{3,4,5,6\}$		$\{1,2,3,4,5,6,7\}$	49
$\{1,2,3,4\},\{3,4,7\},\{1,2,3,4,5,6\}$	—	$\{1,2,3,4,5,6,7\}$	49
$\{1,2,3,4\},\{3,4,7\},\{8,9\}$		$\{1,2,3,4,5,7\},\{8,9\}$	29
$\{1,2,3,4\},\{3,4,5,6\},\{1,2,3,4,5,6\}$	—	$\{1,2,3,4,5,6\}$	36
$\{1,2,3,4\},\{3,4,5,6\},\{8,9\}$		$\{1,2,3,4,5,6\},\{8,9\}$	40
$\{1,2,3,4\},\{1,2,3,4,5,6\},\{8,9\}$	—	$\{1,2,3,4,5,6\},\{8,9\}$	40
$\{3,4,7\},\{3,4,5,6\},\{1,2,3,4,5,6\}$	—	$\{1,2,3,4,5,6,7\}$	49
$\{3,4,7\},\{3,4,5,6\},\{8,9\}$		$\{3,4,5,6,7\},\{8,9\}$	29
$\{3,4,7\},\{1,2,3,4,5,6\},\{8,9\}$	—	$\{1,2,3,4,5,6,7\},\{8,9\}$	53
$\{3,4,5,6\},\{1,2,3,4,5,6\},\{8,9\}$	—	$\{1,2,3,4,5,6\},\{8,9\}$	40
$\{1,2,3,4\},\{3,4,7\},\{3,4,5,6\},\{1,2,3,4,5,6\}$	—	$\{1,2,3,4,5,6,7\}$	49
$\{1,2,3,4\},\{3,4,7\},\{3,4,5,6\},\{8,9\}$	$\mathcal{H}'$	$\{1,2,3,4,5,6,7\},\{8,9\}$	53
$\{1,2,3,4\},\{3,4,7\},\{1,2,3,4,5,6\},\{8,9\}$	—	$\{1,2,3,4,5,6,7\},\{8,9\}$	53
$\{1,2,3,4\},\{3,4,5,6\},\{1,2,3,4,5,6\},\{8,9\}$	—	$\{1,2,3,4,5,6\},\{8,9\}$	40
$\{3,4,7\},\{3,4,5,6\},\{1,2,3,4,5,6\},\{8,9\}$	—	$\{1,2,3,4,5,6,7\},\{8,9\}$	53
$\mathcal{H}$	—	$\{1,2,3,4,5,6,7\},\{8,9\}$	53
$\mathcal{A} \subseteq \mathcal{H}$	$\mathcal{A} \subseteq \mathcal{H}'$	$C_{\mathcal{F}}(N)^*$	$v^{\mathcal{H}'}(\mathcal{A})$

* The unitary feasible coalitions in $C_{\mathcal{F}}(N)$ have been omitted, because the game is zero-normalized.

feasible coalitions of N which each player belongs to. These results were studied by van den Nouweland [10] for the position value in communication situations.

Our first goal, it is to determine the value of the dividends of the conference game $(\mathcal{C}, v^{\mathcal{C}})$ in terms of the dividends of the game (N, v). Note that

$$\pi_i(N, v, \mathcal{F}) = \sum_{C \in \mathcal{C}_i} \frac{1}{|C|} \Phi_C(\mathcal{C}, v^{\mathcal{C}}) = \sum_{C \in \mathcal{C}_i} \frac{1}{|C|} \left[\sum_{\{\mathcal{A} \subseteq \mathcal{C} : C \in \mathcal{A}\}} \frac{\triangle_{v^{\mathcal{C}}}(\mathcal{A})}{|\mathcal{A}|} \right],$$

and the dividends of the conference game are given by

$$\triangle_{v^{\mathcal{C}}}(\mathcal{A}) = \sum_{\mathcal{T} \subseteq \mathcal{A}} (-1)^{|\mathcal{A}| - |\mathcal{T}|} v^{\mathcal{C}}(\mathcal{T}).$$

Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. Consider the associated conference game, we can write,

$$v^{\mathcal{C}} = \sum_{\{\mathcal{H}\subseteq\mathcal{C}:\mathcal{H}\neq\emptyset\}} \triangle_{v^{\mathcal{C}}}(\mathcal{H})u_{\mathcal{H}},$$

where $(\mathcal{C}, u_{\mathcal{H}})$ is the unanimity game defined by

$$u_{\mathcal{H}} : 2^{\mathcal{C}} \to \mathbb{R} \quad \text{with } u_{\mathcal{H}}(\mathcal{A}) = \begin{cases} 1 & \text{if } \mathcal{H} \subseteq \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously,

$$\forall \mathcal{A} \subseteq \mathcal{C},\ v^{\mathcal{C}}(\mathcal{A}) = \sum_{\mathcal{H}\subseteq\mathcal{C}} \triangle_{v^{\mathcal{C}}}(\mathcal{H})u_{\mathcal{H}}(\mathcal{A}) = \sum_{\mathcal{H}\subseteq\mathcal{A}} \triangle_{v^{\mathcal{C}}}(\mathcal{H}).$$

Taking into account that the set $(\mathcal{C}, \subseteq)$ is a partially finite ordered set, we can consider the functions,

$$\triangle_{v^{\mathcal{C}}}, v^{\mathcal{C}} : 2^{\mathcal{C}} \to \mathbb{R}, \text{ with } \triangle_{v^{\mathcal{C}}}(\emptyset) = 0,$$

now, applying Möbius inversion formula (see Stanley [14]),

$$v^{\mathcal{C}}(\mathcal{A}) = \sum_{\mathcal{H}\subseteq\mathcal{A}} \triangle_{v^{\mathcal{C}}}(\mathcal{H}) \Longleftrightarrow \triangle_{v^{\mathcal{C}}}(\mathcal{A}) = \sum_{\mathcal{H}\subseteq\mathcal{A}} \mu(\mathcal{H},\mathcal{A})v^{\mathcal{C}}(\mathcal{H}).$$

In the following theorem is stated that, for certain union stable systems, the dividends from the game $(N, v^{\mathcal{C}})$ can be expressed in terms of the dividends from the game (N, v). We will denote by USI^N the subclass of US^N where the following two conditions are satisfied:

(1) For all $S, T \in \mathcal{F}$, with $|S \cap T| \geq 2$ we have $S \cap T \in \mathcal{F}$.
(2) All non-unitary feasible coalitions can be written in a unique way as a union of non-unitary supports.

For this subclass of union stable cooperation structures that generalize those communication situations for which the graphs do not contain cycles is characterized the position value (see [1]).

If $S \in \mathcal{F}$, we will denote by $\mathcal{S}$ the set of non-unitary supports whose union generate the feasible coalition S and if $S \notin \mathcal{F}$ we will denote by $\overline{\mathcal{S}}$ the set of non-unitary supports corresponding to $\overline{S}$.

Theorem 4.1 *Let* $(N, v, \mathcal{F}) \in USI^N$. *Then*

$$\triangle_{v^{\mathcal{C}}}(\mathcal{A}) = \sum_{S\in V(\mathcal{A})} \triangle_v(S),\ \forall \mathcal{A} \subseteq \mathcal{C},\ \mathcal{A} \neq \emptyset,$$

where $V(\mathcal{A}) = \{S \in 2^N \setminus \{\emptyset\} : \overline{S} \neq \emptyset, \overline{\mathcal{S}} = \mathcal{A}\}$.

Proof. Let $\mathcal{A} \subseteq \mathcal{C}$, $\mathcal{A} \neq \emptyset$. According to the previous reasoning to the theorem, we have

$$\triangle_{v^{\mathcal{C}}}(\mathcal{A}) = \sum_{\mathcal{H}\subseteq\mathcal{A}} \mu(\mathcal{H},\mathcal{A})v^{\mathcal{C}}(\mathcal{H}),$$

and, by definition of the game $v^{\mathcal{C}}$,

$$\triangle_{v^{\mathcal{C}}}(\mathcal{A}) = \sum_{\mathcal{H}\subseteq\mathcal{A}} \mu(\mathcal{H},\mathcal{A}) \sum_{H\in C_{\overline{\mathcal{H}}}(N)} v(H).$$

On the other hand, it is satisfied

$$v(H) = \sum_{S \subseteq H} \triangle_v(S),$$

and, therefore,

$$\triangle_{v^c}(\mathcal{A}) = \sum_{\mathcal{H} \subseteq \mathcal{A}} \mu(\mathcal{H}, \mathcal{A}) \sum_{H \in C_{\overline{\mathcal{H}}}(N)} \left[\sum_{S \subseteq H} \triangle_v(S) \right].$$

Thus,

$$\begin{aligned} \triangle_{v^c}(\mathcal{A}) = \mu(\mathcal{H}_1, \mathcal{A}) & \left[\sum_{S \subseteq H_{11}} \triangle_v(S) + \cdots + \sum_{S \subseteq H_{1k}} \triangle_v(S) \right] + \cdots \\ & + \mu(\mathcal{H}_l, \mathcal{A}) \left[\sum_{S \subseteq H_{l1}} \triangle_v(S) + \cdots + \sum_{S \subseteq H_{lt}} \triangle_v(S) \right] + \cdots \\ & + \mu(\mathcal{H}_p, \mathcal{A}) \left[\sum_{S \subseteq H_{p1}} \triangle_v(S) + \cdots + \sum_{S \subseteq H_{pm}} \triangle_v(S) \right], \end{aligned}$$

where $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_p$ are the elements of the set $\{\mathcal{H} : \mathcal{H} \subseteq \mathcal{A}\}$ and for $i = 1, \ldots, p$, $\{H_{ij}\}_{j \in J}$ are the maximal feasible coalitions of N in $\overline{\mathcal{H}_i}$. The next properties can be noticed:

1. For each $\mathcal{H}_i \in \{\mathcal{H} : \mathcal{H} \subseteq \mathcal{A}\}$, with $1 \leq i \leq p$, the maximal feasible coalitions of N in $\overline{\mathcal{H}_i}$ are disjoint. Therefore $\triangle_v(S)$ appears, at most, only once for each $\mathcal{H}_i \subseteq \mathcal{A}$.
2. Only the coalitions $S \subseteq N$, that are contained in some maximal feasible coalition of N of some $\overline{\mathcal{H}}$ with $\mathcal{H} \subseteq \mathcal{A}$, appear. That is to say, those ones that are contained in some feasible coalition and, therefore, $\underline{S} \neq \emptyset$.
3. If $S \subseteq H_{ij}$ with $H_{ij} \in C_{\overline{\mathcal{H}_i}}(N)$, for some $\mathcal{H}_i$, then $S \subseteq \overline{S} \subseteq H_{ij}$ and, due to the uniqueness in the expression of each feasible coalition as a union of non-unitary supports, we have
$\overline{S} \subseteq H_{ij} \Longleftrightarrow \overline{\mathcal{S}} \subseteq \mathcal{H}_i$.

Taking into account the above remarks,

$$\triangle_{v^c}(\mathcal{A}) = \sum_{\{S \subseteq N : \overline{S} \neq \emptyset\}} \left[\sum_{\overline{\mathcal{S}} \subseteq \mathcal{H} \subseteq \mathcal{A}} \mu(\mathcal{H}, \mathcal{A}) \right] \triangle_v (S).$$

Moreover,

1. If $\overline{\mathcal{S}} = \mathcal{A}$, we have

$$\sum_{\overline{\mathcal{S}} \subseteq \mathcal{H} \subseteq \mathcal{A}} \mu(\mathcal{H}, \mathcal{A}) = \mu(\mathcal{A}, \mathcal{A}) = 1.$$

2. If $\overline{\mathcal{S}} \neq \mathcal{A}$, the set $\overline{\mathcal{S}} \subseteq \mathcal{H} \subseteq \mathcal{A}$ is the interval $[\overline{\mathcal{S}}, \mathcal{A}]$ which is a finite lattice with, at least, two elements and hence [14, Corollary 3.9.3] it follows

$$\sum_{\{\mathcal{H}:\mathcal{H}\wedge\overline{\mathcal{S}}=\overline{\mathcal{S}}\}} \mu(\mathcal{H},\mathcal{A}) = \sum_{\{\mathcal{H}:\overline{\mathcal{S}}\subseteq\mathcal{H}\subseteq\mathcal{A}\}} \mu(\mathcal{H},\mathcal{A}) = 0.$$

So, we conclude

$$\triangle_{v^{\mathcal{C}}}(\mathcal{A}) = \sum_{\{S\subseteq N:\overline{S}\neq\emptyset,\overline{S}=\mathcal{A}\}} \triangle_v(S), \text{ with } \mathcal{A}\neq\emptyset. \qquad \square$$

Next, we compute the position value for union stable systems by the $\mathcal{F}$-components of N. First, some relations between $\mathcal{F}$-components, feasible coalitions, and supports of a union stable system are given.

Proposition 4.2 *Let $\mathcal{F}$ be a union stable system and $B(\mathcal{F})$ its basis. Then*

(a) *If $N\notin\mathcal{F}$, we define the partition $\{\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_p\}$ of the basis $B(\mathcal{F})$ by $\mathcal{B}_i = \{B\in B(\mathcal{F}) : B\subseteq N_i, N_i\in C_{\mathcal{F}}(N)\}$. Then, for all $B\in\mathcal{B}_i$, $B'\in\mathcal{B}_j$, with $i\neq j$, $1\leq i,j\leq p$, we have $B\cap B'=\emptyset$.*
(b) *Let $\mathcal{I}\subseteq B(\mathcal{F})$, $\mathcal{J}\subseteq B(\mathcal{F})$ such that for all $B\in\mathcal{I}$ and for all $B'\in\mathcal{J}$, we have $B\cap B'=\emptyset$. Then*

$$C_{\overline{\mathcal{I}\cup\mathcal{J}}}(N) = C_{\overline{\mathcal{I}}}(N)\cup C_{\overline{\mathcal{J}}}(N).$$

The above properties are used in the proof of the following theorem.

Theorem 4.3 *Let $(N,v,\mathcal{F})$ be a union stable cooperation structure. Let $i\in N$ and $M\in C_{\mathcal{F}}(N)$ such that $i\in M$. Then*

$$\pi_i(N,v,\mathcal{F}) = \pi_i(M,v|_M,\mathcal{F}_M),$$

where $v|_M$ is the restriction of v to *M and $\mathcal{F}_M = \{F\in\mathcal{F} : F\subseteq M\}$.*

Proof. By definition, we have

$$\pi_i(N,v,\mathcal{F}) = \sum_{C\in\mathcal{C}_i} \frac{1}{|C|}\Phi_C(\mathcal{C},v^{\mathcal{C}}).$$

If $i\in M$, with $M\in C_{\mathcal{F}}(N)$, Proposition 4.2 (a) states that there exists a partition $\{\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_p\}$ of $\mathcal{C}$, where $\mathcal{D}_k$, $k=1,\ldots,p$ is the collection of non-unitary supports contained in M and such that M is the union of them. Hence, $\mathcal{C}_i\subseteq\mathcal{D}_k$, for a unique k, and

$$\pi_i(N,v,\mathcal{F}) = \sum_{\{D\in\mathcal{D}_k:i\in D\}} \frac{1}{|D|}\Phi_D(\mathcal{C},v^{\mathcal{C}}).$$

Next, we establish that $\Phi_D(\mathcal{C},v^{\mathcal{C}}) = \Phi_D(\mathcal{D}_k,v^{\mathcal{D}_k})$ for any $D\in\mathcal{D}_k$. By definition of the Shapley value

$$\Phi_D(\mathcal{C},v^{\mathcal{C}}) = \sum_{\{\mathcal{S}\subseteq\mathcal{C}:D\notin\mathcal{S}\}} \gamma(\mathcal{S})\left[v^{\mathcal{C}}(\mathcal{S}\cup\{D\}) - v^{\mathcal{C}}(\mathcal{S})\right]$$

and, for $\mathcal{S}\subseteq\mathcal{C}$, $v^{\mathcal{C}}(\mathcal{S}) = v^{\overline{\mathcal{S}}}(N) = \sum_{H\in C_{\overline{\mathcal{S}}}(N)} v(H)$. Taking into account part (b) of Proposition 4.2, we obtain

$$C_{\overline{\mathcal{S}}}(N) = C_{\overline{\mathcal{S}\cap\mathcal{C}}}(N) = C_{\overline{\mathcal{S}\cap\left(\mathcal{D}_1\cup\mathcal{D}_2\cup\ldots\cup\mathcal{D}_p\right)}}(N)$$
$$= C_{\overline{(\mathcal{S}\cap\mathcal{D}_1)\cup(\mathcal{S}\cap\mathcal{D}_2)\cup\ldots\cup\left(\mathcal{S}\cap\mathcal{D}_p\right)}}(N)$$
$$= C_{\overline{(\mathcal{S}\cap\mathcal{D}_1)}}(N)\cup C_{\overline{(\mathcal{S}\cap\mathcal{D}_2)}}(N)\cup\ldots\cup C_{\overline{\left(\mathcal{S}\cap\mathcal{D}_p\right)}}(N),$$

and applying it to $v^{\mathcal{C}}(\mathcal{S})$ and $v^{\mathcal{C}}(\mathcal{S}\cup\{D\})$, we have

$$v^{\mathcal{C}}(\mathcal{S}\cup\{D\}) - v^{\mathcal{C}}(\mathcal{S}) = \sum_{H\in C_{\overline{\mathcal{S}\cup\{D\}}}(N)} v(H) - \sum_{H\in C_{\overline{\mathcal{S}}}(N)} v(H)$$
$$= \sum_{H\in C_{\overline{(\mathcal{S}\cup\{D\})\cap\mathcal{D}_k}}(N)} v(H) + \sum_{\left\{H:H\in C_{\overline{(\mathcal{S}\cup\{D\})\cap\mathcal{D}_t}}(N),t\neq k\right\}} v(H)$$
$$- \sum_{H\in C_{\overline{\mathcal{S}\cap\mathcal{D}_k}}(N)} v(H) - \sum_{\left\{H:H\in C_{\overline{\mathcal{S}\cap\mathcal{D}_t}}(N),t\neq k\right\}} v(H)$$
$$= \sum_{H\in C_{\overline{(\mathcal{S}\cup\{D\})\cap\mathcal{D}_k}}(N)} v(H) - \sum_{H\in C_{\overline{\mathcal{S}\cap\mathcal{D}_k}}(N)} v(H)$$
$$= v^{\overline{(\mathcal{S}\cap\mathcal{D}_k)\cup\{D\}}}(N) - v^{\overline{\mathcal{S}\cap\mathcal{D}_k}}(N),$$

since, if $t\neq k$,

$$C_{\overline{(\mathcal{S}\cup\{D\})\cap\mathcal{D}_t}}(N) = C_{\overline{\mathcal{S}\cap\mathcal{D}_t}}(N),$$

because $\mathcal{D}_t\cap\{D\}=\emptyset$ and $(\mathcal{S}\cup\{D\})\cap\mathcal{D}_k = (\mathcal{S}\cap\mathcal{D}_k)\cup\{D\}$.

Moreover, if $\mathcal{S}$ covers all support coalitions of $\mathcal{C}$ which the non-unitary support D does not belong to, then $\mathcal{S}\cap\mathcal{D}_k$ covers all non-unitary support coalitions contained in $\mathcal{D}_k$ in which the support D is not contained. Hence,

$$\Phi_D\left(\mathcal{C},v^{\mathcal{C}}\right) = \sum_{\{\mathcal{S}\subseteq\mathcal{C}:D\notin\mathcal{S}\}} \gamma(\mathcal{S})\left[v^{\mathcal{C}}(\mathcal{S}\cup\{D\}) - v^{\mathcal{C}}(\mathcal{S})\right]$$

$$= \sum_{\{\mathcal{T}\subseteq\mathcal{D}_k:D\notin\mathcal{T}\}}\left[\sum_{\{\mathcal{S}\subseteq\mathcal{C}:D\notin\mathcal{S},\ \mathcal{S}\cap\mathcal{D}_k=\mathcal{T}\}}\gamma(\mathcal{S})\right]\left[v^{\overline{\mathcal{T}\cup\{D\}}}(N) - v^{\overline{\mathcal{T}}}(N)\right]$$

$$= \sum_{\{\mathcal{T}\subseteq\mathcal{D}_k:D\notin\mathcal{T}\}}\left[\sum_{\{\mathcal{S}\subseteq\mathcal{C}:D\notin\mathcal{S},\ \mathcal{S}\cap\mathcal{D}_k=\mathcal{T}\}}\gamma(\mathcal{S})\right]\left[v^{\mathcal{D}_k}(\mathcal{T}\cup\{D\}) - v^{\mathcal{D}_k}(\mathcal{T})\right].$$

Using a reasoning completely similar to the one given for the Myerson value in [2, Theorem 4.2], we obtain

$$\sum_{\{\mathcal{S}\subseteq\mathcal{C}:D\notin\mathcal{S},\ \ \mathcal{S}\cap\mathcal{D}_k=\mathcal{T}\}}\gamma(\mathcal{S}) = \gamma(\mathcal{T}),$$

and, hence

$$\Phi_D\left(\mathcal{C},v^{\mathcal{C}}\right) = \Phi_D\left(\mathcal{D}_k,v^{\mathcal{D}_k}\right),\ \text{for all } D\in\mathcal{D}_k.$$

□

The above theorem shows that in order to compute the position value of a player it suffices to consider the component of N containing this particular player.

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