International Journal of Foundations of Computer Science

ON THE COMPLEXITY OF COMPUTING VALUES OF RESTRICTED GAMES

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Received (received date)
Revised (revised date)
Communicated by Editor's name

ABSTRACT

The aim of this paper is to compute Shapley's and Banzhaf's values of cooperative games restricted by a combinatorial structure. There have been previous models developed to study the problem of games with partial cooperation. Games restricted by a communication graph were introduced by Myerson and Owen. Another type of combinatorial structure introduced by Gilles, Owen and van den Brink is equivalent to a subclass of antimatroids. Cooperative games in which the set of players is a partially ordered set, that is, games on distributive lattices was investigated by Faigle and Kern. We introduce a new combinatorial structure called augmenting system which is a generalization of the antimatroid structure and the system of connected subgraphs of a graph. We present new algorithmic procedures for computing values of games under augmenting systems restrictions and we show that there exist problems with polynomial algorithm complexity.

Keywords: Complexity, Cooperative games, Shapley value, Banzhaf value.

1. Introduction

Cooperative games under combinatorial restrictions are cooperative games in which the players have restricted communication possibilities, which are defined by a combinatorial structure. The first model in which the restrictions are defined by the connected subgraphs of a graph is introduced by Myerson [11]. Since then, many other situations where players have communication restrictions have been studied in cooperative game theory. Contributions on graph-restricted games include Owen [12], Borm, Owen, and Tijs [3] and Hamiache [8]. In these models the possibilities of coalition formation are determined by the positions of the players in a *communication graph.* Another type of asymmetry among the players is introduced in Gilles, Owen, and van den Brink [7]. This line of research focuses on the possibilities of coalition formation determined by the positions of the players in the so-called *per-*

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mission structure. A model in which cooperation possibilities in a game are limited by a partial ordering of the set of players can be found in Faigle and Kern [6].

In the present paper, we use the restricted cooperation model derived from a combinatorial structure called *augmenting system.* Section 2 introduces this structure which is a generalization of the antimatroid structure and the system of connected subgraphs of a graph. Furthermore, this new set system includes the conjunctive and disjunctive systems derived from a permission structure. Section 3 is devoted to the games under augmenting systems which generalize the ones studied on graphs and permission structures. Using the structural properties from these systems we will be able to compute the Shapley and Banzhaf values for games under augmenting systems restrictions. These values are computed by means of the original game without having to calculate the restricted game and taking into account only the coalitions in the augmenting system. Section 4 introduces the computational complexity and Sections 5 and 6 present our results about the complexity of computing the Shapley and Banzhaf values of restricted games.

2. Augmenting systems

Antimatroids were introduced by Dilworth [4] as particular examples of semimodular lattices. Since then, several authors have obtained the same concept by abstracting various combinatorial situations (see Korte, Lovász, and Schrader [10]). In this section, a general cooperation structure is introduced, which is a weakening of the antimatroid structure.

Let N be a finite set. A *set system* over N is a pair $(N, \mathcal{F})$ where $\mathcal{F} \subseteq 2^N$ is a family of subsets. The sets belonging to $\mathcal{F}$ are called *feasible.* We will write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$ respectively.

Definition 1 *A set system $(N, \mathcal{A})$ is an antimatroid if*

- *A1.* $\emptyset \in \mathcal{A}$,
- *A2. for* $S, T \in \mathcal{A}$ *we have* $S \cup T \in \mathcal{A}$,
- *A3. for* $S \in \mathcal{A}$ *with* $S \neq \emptyset$, *there exists* $i \in S$ *such that* $S \setminus i \in \mathcal{A}$.

The definition of antimatroid implies the following *augmentation property*: If $S, T \in \mathcal{A}$ with $|T| > |S|$ then there exists $i \in T \setminus S$ such that $S \cup i \in \mathcal{A}$. We call a set system $(N, \mathcal{F})$ *normal* if

$$N = \bigcup_{S \in \mathcal{F}} S.$$

If $(N, \mathcal{A})$ is a normal antimatroid then property A2 implies that $N \in \mathcal{A}$.

Definition 2 *An augmenting system is a normal set system $(N, \mathcal{F})$ with the following properties:*

- *P1.* $\emptyset \in \mathcal{F}$,
- *P2. for* $S, T \in \mathcal{F}$ *with* $S \cap T \neq \emptyset$, *we have* $S \cup T \in \mathcal{F}$,

P3. for $S, T \in \mathcal{F}$ with $S \subset T$, there exists $i \in T \setminus S$ such that $S \cup i \in \mathcal{F}$.

Proposition 1 *An augmenting system $(N, \mathcal{F})$ is an antimatroid if and only if $\mathcal{F}$ is closed under union.*

Proof. The necessary condition follows from A2. Conversely, we only have to prove A3. Let $S \in \mathcal{F}$ with $S \neq \emptyset$. By property P3 there exists a chain of feasible subsets $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{s-1} \subset S_s = S$, such that $S_k \in \mathcal{F}$ and $|S_k| = k$, for $0 \leq k \leq s$. Hence there exists an element $i \in S$ such that $S \setminus i = S_{s-1} \in \mathcal{F}$. □

Remark. Notice that normal antimatroids are always augmenting systems.

Example 1 *The following collections of subsets of N, given by $\mathcal{F} = 2^N$ and $\mathcal{F} = \{\emptyset, \{1\}, \ldots, \{n\}\}$, are the maximum augmenting system and a minimal augmenting system over N, respectively.*

Example 2 *In a communication graph $G = (N, E)$, the set system $(N, \mathcal{F})$ given by $\mathcal{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } G\}$, is an augmenting system.*

Example 3 *Gilles et al. [7] showed that the feasible coalitions system $(N, \mathcal{F})$ derived from the conjunctive or disjunctive approach contains the empty set, the ground set N, and that it is closed under union. Algaba et al. [1] showed that the coalitions systems derived from both approaches were identified to poset antimatroids and antimatroids with the path property, respectively. Therefore, these coalitions systems are augmenting systems.*

Convex geometries are a combinatorial abstraction of convex sets introduced by Edelman and Jamison [5].

Definition 3 *A set system $(N, \mathcal{G})$ is a convex geometry if it satisfies the following properties:*

C1. $\emptyset \in \mathcal{G}$,

C2. for $S, T \in \mathcal{G}$ we have $S \cap T \in \mathcal{G}$,

C3. for $S \in \mathcal{G}$ with $S \neq N$, there exists $i \in N \setminus S$ such that $S \cup i \in \mathcal{G}$.

Proposition 2 *An augmenting system $(N, \mathcal{F})$ is a convex geometry if and only if $\mathcal{F}$ is closed under intersection and $N \in \mathcal{F}$.*

Proof. The necessary conditions follow from properties C2 and C3. To prove sufficiency, note that $(N, \mathcal{F})$ satisfies C1 and C2, i.e., it is a closure system over N. Moreover, $(N, \mathcal{F})$ satisfies the weak augmentation property P3 and $N \in \mathcal{F}$. Then for every $S \in \mathcal{F}$ with $S \neq N$, there exists $i \in N \setminus S$ such that $S \cup i \in \mathcal{F}$. □

Definition 4 *Let $(N, \mathcal{F})$ be an augmenting system. For a feasible coalition $S \in \mathcal{F}$, we define the set $S^* = \{i \in N \setminus S : S \cup i \in \mathcal{F}\}$ of augmentations of S and the set $S^+ = S \cup S^* = \{i \in N : S \cup i \in \mathcal{F}\}$.*

Proposition 3 *Let $(N, \mathcal{F})$ be an augmenting system. Then the interval*

$$\left[S, S^{+}\right]_{\mathcal{F}} = \left\{C \in \mathcal{F} : S \subseteq C \subseteq S^{+}\right\}$$

is a Boolean algebra for every non empty $S \in \mathcal{F}$.

Proof. It is suffices to show that $[S, S^{+}]_{\mathcal{F}} = \{C \subseteq N : S \subseteq C \subseteq S^{+}\}$, i.e. for every $C \subseteq N$ such that $S \subseteq C \subseteq S^{+}$ we have $C \in \mathcal{F}$. If $S^{*} = \emptyset$ then $[S, S^{+}]_{\mathcal{F}} = \{S\}$. Otherwise $S^{*} = \{i_1, \ldots, i_p\}$ and $S \subseteq C \subseteq S^{+}$ implies $C = S \cup \{i_1, \ldots, i_q\}$ for some $1 \leq q \leq p$. We prove that $C \in \mathcal{F}$ by induction on q. For $q = 1$ we know that $S \cup \{i_1\} \in \mathcal{F}$. Assume $S \cup \{i_1, \ldots, i_k\} \in \mathcal{F}$. Since $S \cup \{i_{k+1}\} \in \mathcal{F}$ and $(S \cup \{i_1, \ldots, i_k\}) \cap (S \cup \{i_{k+1}\}) = S \neq \emptyset$, property P2 yields $S \cup \{i_1, \ldots, i_{k+1}\} \in \mathcal{F}$. □

Let $(N, \mathcal{F})$ be a set system and let $S \subseteq N$ be a subset. A feasible subset $C \in \mathcal{F}$ with $C \subseteq S$ is called a *basis* of S if $C \cup i \notin \mathcal{F}$ for all $i \in S \setminus C$. The maximal non empty feasible subsets of S are called *components* of S. Clearly, every component of S is a basis of S. However, the converse is not true, as the following example shows:

Example 4 *If $N = \{1, 2, 3\}$ and $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{2, 3\}, N\}$ then $C = \{1\}$ is a basis of N, but the only component of N is the ground set N.*

Observe that if $(N, \mathcal{A})$ is an antimatroid then any subset $S \subseteq N$ has a unique basis given by the following operator $\operatorname{int}(S) = \bigcup \{C \in \mathcal{A} : C \subseteq S\}$. This feasible set is also the unique component of S.

Proposition 4 *Let $(N, \mathcal{F})$ be an augmenting system and let $S \subseteq N$ be a subset. Then a non empty feasible subset $C \subseteq S$ is a basis of S if and only if C is a component of S.*

Proof. Let $C \in \mathcal{F}$ be a basis of S and suppose C is not a component of S, i.e. there exists $D \in \mathcal{F}$ such that $C \subset D \subseteq S$. Then in view property P3 there exists an element $i \in D \setminus C \subseteq S \setminus C$ such that $C \cup i \in \mathcal{F}$, which is a contradiction. □

We denote by $C_{\mathcal{F}}(S)$ the set of the components of a subset $S \subseteq N$. Observe that the set $C_{\mathcal{F}}(S)$ may be the empty set. This set will play a role in the concept of game restricted by an augmenting system. We close this section proving a property of the set $C_{\mathcal{F}}(S)$.

Proposition 5 *A set system $(N, \mathcal{F})$ satisfies property P2 if and only if for any $S \subseteq N$ with $C_{\mathcal{F}}(S) \neq \emptyset$, the components of S form a partition of a subset of S.*

Proof. We suppose that $(N, \mathcal{F})$ satisfies P2 and let S_1, S_2 be components of S. If $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2 \in \mathcal{F}$ and we have that $S_i \subset S_1 \cup S_2 \subseteq S$ for $i \in \{1, 2\}$. This contradicts the fact that S_1 and S_2 are components of S. Conversely, assume for any S with $C_{\mathcal{F}}(S) \neq \emptyset$, that its components form a partition of a subset of S. Suppose that $(N, \mathcal{F})$ do not satisfies P2. Then there are $A, B \in \mathcal{F}$, with $A \cap B \neq \emptyset$ and $A \cup B \notin \mathcal{F}$. Hence, there must be a component $C_1 \in C_{\mathcal{F}}(A \cup B)$, with $A \subseteq C_1$ and a component $C_2 \in C_{\mathcal{F}}(A \cup B)$, with $B \subseteq C_2$ such that $C_1 \neq C_2$. This contradicts the fact that the components of $A \cup B$ are disjoint. □

3. Games restricted by augmenting systems

Definition 5 *Let $v : 2^N \to \mathbb{R}$ be a cooperative game and let $(N, \mathcal{F})$ be an augmenting system. The restricted game $v^{\mathcal{F}} : 2^N \to \mathbb{R}$, is defined by*

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T).$$

Remark. If $(N, \mathcal{F})$ is the augmenting system given by the connected subgraphs of a graph $G = (N, E)$, then the game $v^{\mathcal{F}}$ is a graph-restricted game which is studied by Myerson [11], Owen [12], Potters and Reijnierse [13].

If $S \in \mathcal{F}$ then $v^{\mathcal{F}}(S) = v(S)$. Let us denote by Γ^N the vector space of all cooperative games (N, v), i.e. functions $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Every cooperative game (N, v) is uniquely determined by the collection of its values $\{v(S) : S \subseteq N,\ S \neq \emptyset\}$. Then the space Γ^N will be identified with $\mathbb{R}^{2^n - 1}$. For any $S \subseteq N,\ S \neq \emptyset$, we define the *unanimity* game

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

Every game is a unique linear combination of unanimity games (cf. [14]),

$$v = \sum_{S \subseteq N} d_S u_S, \quad \text{where } d_S = \sum_{T \subseteq S} (-1)^{|S| - |T|} v(T).$$

We shall call d_S the *dividend* of S in the game v. Owen [12] showed the following property: The unanimity games u_S, where S is connected in the graph G, form a basis of the graph-restricted games.

Let $(N, \mathcal{F})$ be the system of connected subgraphs of a graph $G = (N, E)$. Then Hamiache [8, Lemma 2] proved a formula for computing the dividends in the game $v^{\mathcal{F}}$ by using the values in the original game v. Next, we extend the Hamiache's formula to the case when $(N, \mathcal{F})$ is an augmenting system.

Proposition 6 *Let $(N, \mathcal{F})$ be an augmenting system and let (N, v) be a game. Then the restricted game $v^{\mathcal{F}} = \sum_{C \in \mathcal{F}} d_C u_C$ where the dividend*

$$d_C = \sum_{\{S \in \mathcal{F} \,:\, S \subseteq C \subseteq S^+\}} (-1)^{|C| - |S|} v(S),$$

for every non empty $C \in \mathcal{F}$, and $d_C = 0$ otherwise.

Proof. The game $v^{\mathcal{F}}$ satisfies for every $C \subseteq N$,

$$v^{\mathcal{F}}(C) = \sum_{T \subseteq N} d_T u_T(C) = \sum_{T \subseteq C} d_T,$$

where d_T the dividend of T in the game $v^{\mathcal{F}}$. Then, the Möbius inversion formula implies (see Stanley [15, p. 116]) that

$$d_C = \sum_{T \subseteq C} (-1)^{|C| - |T|} v^{\mathcal{F}}(T).$$

It follows from $v^{\mathcal{F}}(\emptyset) = 0$ that $d_\emptyset = 0$. So we may assume that $C \neq \emptyset$. The definition of $v^{\mathcal{F}}$ implies that

$$\begin{aligned} d_C &= \sum_{T \subseteq C} (-1)^{|C|-|T|} \left(\sum_{S \in C_{\mathcal{F}}(T)} v(S) \right) \\ &= \sum_{\{S \in \mathcal{F} \,:\, S \subseteq C\}} \left(\sum_{\{T \subseteq C \,:\, S \in C_{\mathcal{F}}(T)\}} (-1)^{|C|-|T|} \right) v(S). \end{aligned}$$

Let $S \in \mathcal{F}$ with $S \subseteq T$. We show first that

$$\{T \subseteq C : S \in C_{\mathcal{F}}(T)\} = \left\{T \subseteq C : T \setminus S \subseteq C \setminus S^+\right\}.$$

We take $T \subseteq C$. If $S \in C_{\mathcal{F}}(T)$ then by Proposition 4, S is a basis of T and hence the set of its augmentations S^* satisfies $S^* \cap T = \emptyset$. Then for each $i \in T \setminus S$ we have $i \in C$ and $i \notin S \cup S^* = S^+$.

Conversely, let $T \subseteq C$ be a set such that $T \setminus S \subseteq C \setminus S^+$. Then for each $i \in T \setminus S$ we have $i \notin S^+$ and hence $S \cup i \notin \mathcal{F}$. Thus, the feasible set S is a basis of T and we conclude that $S \in C_{\mathcal{F}}(T)$.

Therefore, the coefficients of d_C satisfy

$$\begin{aligned} \sum_{\{T \subseteq C \,:\, S \in C_{\mathcal{F}}(T)\}} (-1)^{|C|-|T|} &= \sum_{\{T \subseteq C \,:\, S \subseteq T,\, T \setminus S \subseteq C \setminus S^+\}} (-1)^{|C|-|T|} \\ &= (-1)^{|C|-|S|} \left(\sum_{R \subseteq C \setminus S^+} (-1)^{-|R|} \right). \end{aligned}$$

Next, we compute

$$\sum_{R \subseteq C \setminus S^+} (-1)^{-|R|} = \sum_{R \subseteq C \setminus S^+} (-1)^{|R|} = (1-1)^{\left|C \setminus S^+\right|} = \begin{cases} 1 & \text{if } C \setminus S^+ = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $C \setminus S^+ = \emptyset \Leftrightarrow C \subseteq S^+$, and hence

$$\begin{aligned} d_C &= \sum_{\{S \in \mathcal{F} \,:\, S \subseteq C,\, C \setminus S^+ = \emptyset\}} (-1)^{|C|-|S|} v(S) \\ &= \sum_{\{S \in \mathcal{F} \,:\, S \subseteq C \subseteq S^+\}} (-1)^{|C|-|S|} v(S). \end{aligned}$$

To complete the proof we observe that Proposition 3 implies that the set $C \in \mathcal{F}$. Otherwise $C \setminus S^+ \neq \emptyset$, and so $d_C = 0$ for all $C \notin \mathcal{F}$. □

Let (N, v) be a game and let $(N, \mathcal{F})$ be a augmenting system. The *Shapley value* for player i in the restricted game $v^{\mathcal{F}}$ is given by $\Phi_i\left(N, v^{\mathcal{F}}\right)$, for all $i \in N$. The *Banzhaf value* for player i in game $v^{\mathcal{F}}$ is given by $\beta'_i\left(N, v^{\mathcal{F}}\right)$, for all $i \in N$.

The Shapley and Banzhaf values are linear mappings with respect to the characteristic function and the images of the unanimity games are respectively (cf. [12]):

$$\Phi_i(N, u_S) = \begin{cases} 1/|S| & \text{if } i \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta'_i(N, u_S) = \begin{cases} 1/2^{|S \setminus i|} & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of dividends d_S in game $v^{\mathcal{F}}$, we have that

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \subseteq N : i \in S\}} \frac{d_S}{|S|} = \sum_{\{S \in \mathcal{F} : i \in S\}} \frac{d_S}{|S|}, \tag{1}$$

$$\beta'_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \subseteq N : i \in S\}} \frac{d_S}{2^{|S \setminus i|}} = \sum_{\{S \in \mathcal{F} : i \in S\}} \frac{d_S}{2^{|S \setminus i|}}. \tag{2}$$

Bilbao [2] showed the following *explicit formulas,* in terms of v, for the Shapley and Banzhaf values of the players in the restricted game $v^{\mathcal{F}}$.

Theorem 1 *Let $(N, \mathcal{F})$ be an augmenting system and let (N, v) be a game. Then*

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{T \in \mathcal{F} : i \in T\}} \frac{(t-1)!\, t^*!}{t^+!} v(T) - \sum_{\{T \in \mathcal{F} : i \in T^*\}} \frac{t!\, (t^*-1)!}{t^+!} v(T),$$

$$\beta'_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{T \in \mathcal{F} : i \in T\}} \frac{1}{2^{t^+-1}} v(T) - \sum_{\{T \in \mathcal{F} : i \in T^*\}} \frac{1}{2^{t^+-1}} v(T),$$

where $t = |T|$, $t^* = |T^*|$ and $t^+ = |T^+|$.

Remark. Notice that if $\mathcal{F} = 2^N$, then $T^* = N \setminus T$, and $T^+ = N$ for every $T \in \mathcal{F}$. Thus, the formulas obtained in the above theorem are equal to the classical Shapley and Banzhaf values for the game v (see [14, p. 35]).

Let us consider a set system $(N, \mathcal{F})$. An element i of a feasible set $S \in \mathcal{F}$ is an *extreme point* of S if $S \setminus i \in \mathcal{F}$. The set of extreme points of S is denoted by $\text{ex}(S)$. The formulas for computing the Shapley and Banzhaf values of the players in the restricted game $v^{\mathcal{F}}$ can be further simplified when the player is an extreme point of every feasible coalition.

Theorem 2 *Let $(N, \mathcal{F})$ be an augmenting system and let (N, v) be a game such that $v(i) = 0$ for all $i \in N$. If $i \in ex(S)$ for all $S \in \mathcal{F}$ which contains i, then*

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \in \mathcal{F} : i \in S,\ |S|>1\}} \frac{(s-1)!\, s^*!}{s^+!} \left[v(S) - v(S \setminus i)\right],$$

$$\beta'_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \in \mathcal{F} : i \in S,\ |S|>1\}} \frac{1}{2^{s^+-1}} \left[v(S) - v(S \setminus i)\right],$$

where $s = |S|$, $s^* = |S^*|$ and $s^+ = |S^+|$.

Remark. Let $(N,\mathcal{F})$ be an augmenting system which is a convex geometry. Then for every $i \in \mathrm{ex}\,(N)$ we have $S \setminus i = (N \setminus i) \cap S \in \mathcal{F}$ for all $S \in \mathcal{F}$ such that $i \in S$. Hence, if $i \in \mathrm{ex}\,(N)$ then $i \in \mathrm{ex}\,(S)$ for all $S \in \mathcal{F}$ with $i \in S$.

Example 5 *The augmenting system of the connected subgraphs of a path P_n is $\mathcal{F} = \{[i,j] : 1 \leq i \leq j \leq n\} \cup \{\emptyset\}$ and $ex(N) = \{1, n\}$.*

Example 6 *Let $K_{1,n-1}$ be a star on n vertices and let 1 be the center of star. The augmenting system of the connected subgraphs of $K_{1,n-1}$ is given by $\mathcal{F} = \{S \subseteq N : 1 \in S \ \text{ or } \ |S| = 1\} \cup \{\emptyset\}$. Then $ex(N) = \{2, \ldots, n\}$, and for all $S \in \mathcal{F}$ such that $|S| > 1$, we infer that $1 \in S$, $S^* = N \setminus S$ and $S^+ = N$. Moreover, the set $\{S \in \mathcal{F} : 1 \in S^*, \ |S| > 1\} = \emptyset$.*

Using the above properties, one can derive from Theorems 1 and 2 the following result:

Theorem 3 *Let $(N,\mathcal{F})$ be an augmenting system of the connected subgraphs of the star $K_{1,n-1}$ and let (N,v) be a game. If (N,v) is a game such that $v(i) = 0$ for all $i \in N$, then*

$$\begin{aligned} \Phi_1\left(N, v^{\mathcal{F}}\right) &= \sum_{\{S \in \mathcal{F} : \, 1 \in S, \ |S| > 1\}} \frac{(s-1)!(n-s)!}{n!} v(S), \\ \Phi_i\left(N, v^{\mathcal{F}}\right) &= \sum_{\{S \in \mathcal{F} : \, i \in S, \ |S| > 1\}} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v(S \setminus i)\right], \end{aligned}$$

for $i \in \{2, \ldots, n\}$.

4. Computational complexity

The *time complexity* function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ of an algorithm A is the maximal number $f(n)$ of iterations of a universal Turing machine makes before halting, taken over all inputs of size n. We say that an algorithm has *space complexity* at most $f(n)$, if it can be computed by a Turing machine with space demand (cells and tapes) at most $f(n)$.

Let f and g be functions from $\mathbb{Z}_+$ to $\mathbb{Z}_+$. We write $f(n) = O(g(n))$, in words *f is of the order of g*, if there are positive integers c and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$. We write $f(n) = \Omega(g(n))$ if the opposite happens, that is, $g(n) = O(f(n))$. If f and g have exactly the same rate of growth, then we write $f(n) = \Theta(g(n))$. For instance, if $p(n)$ is a polynomial of degree d, then $p(n) = \Theta\left(n^d\right)$. The above $O\Omega\Theta$-notation was proposed by Knuth [9].

We analyze our algorithms in the *arithmetic model*, that is, we count elementary arithmetic operations and assignments. For instance, the standard algorithm for computing the product of two $n \times n$ matrices is $O\left(n^3\right)$.

The programs of our language contain only *assignments* and a **for**-*loop construct*. We use the symbol $\leftarrow$ for assignments, for example, $g(x) \leftarrow 1$ denotes setting the value of $g(x)$ to 1. A **for**-loop to calculate $\sum_{i \in I} a_i$, can be defined by

$h \leftarrow 0$
for $i \in I$ **do**
$h \leftarrow h + a_i$
endfor

5. The complexity of computing dividends

We study the complexity of computing the Shapley and Banzhaf values of the restricted game $\left(N, v^{\mathcal{F}}\right)$ by using dividends and formulas (1) and (2).

Proposition 7 *Let $(N, \mathcal{F})$ be an augmenting system and let (N, v) be a game. Computing the dividends of the restricted game $\left(N, v^{\mathcal{F}}\right)$ requires a time $O\left(3^n\right)$ and a space $\Omega\left(|\mathcal{F}|\right)$.*

Proof. Proposition 6 implies the following algorithm:

Algorithm *dividend* $\left(N, v^{\mathcal{F}}\right)$
$d_{\emptyset} \leftarrow 0$

for $i \in \mathbb{Z}$ **with** $1 \leq i \leq n$ **do**

for $j \in \mathbb{Z}$ **with** $1 \leq j \leq T(i)$ **do**

$d_{T_i^j} \leftarrow \sum_{\{S \in \mathcal{F}: S \subseteq T_i^j \subseteq S^+\}} (-1)^{\left|T_i^j\right| - |S|} v(S)$

endfor

endfor

where T_i^j is the j-th feasible coalition of size i and $T(i)$ is the number of feasible coalitions of cardinality i. Observe that $T(i) \leq \binom{n}{i}$ for $i = 1, \ldots, n$. The number of non empty feasible coalitions contained in T_i^j is denoted by $C(i) \leq 2^i - 1$. Also note that it is suffices to calculate dividends of the feasible coalitions. The execution time of the *dividend* algorithm satisfies:

$$
\begin{aligned}
t(div) &= 1 + t(loop1) = 1 + \sum_{i=1}^{n} t(loop2) \\
&= 1 + \sum_{i=1}^{n} \sum_{j=1}^{T(i)} t(assignment) = 1 + \sum_{i=1}^{n} \sum_{j=1}^{T(i)} (1 + t(sum)) \\
&\leq 1 + \sum_{i=1}^{n} \sum_{j=1}^{T(i)} \left(1 + \sum_{k=1}^{C(i)} 2\right) \leq 1 + \sum_{i=1}^{n} \left(1 + 2\left(2^i - 1\right)\right) T(i) \\
&\leq 1 + \sum_{i=1}^{n} \left(1 + 2\left(2^i - 1\right)\right) \binom{n}{i} = 1 + 2 \sum_{i=1}^{n} \binom{n}{i} 2^i - \sum_{i=1}^{n} \binom{n}{i} \\
&= 1 + 2\left(3^n - 1\right) - \left(2^n - 1\right) = 2\left(3^n - 2^{n-1}\right).
\end{aligned}
$$

Therefore, the time complexity of *dividend* is $O(3^n)$. On the other hand, if it is taken into account that the computation of the dividends is by an ascending process which requires to keep the dividends of each one of the feasible coalitions, it is obtained that the required space is $\Omega(|\mathcal{F}|)$. □

Proposition 8 *Let $(N, \mathcal{F})$ be an augmenting system and let (N, v) be a game. Computing the Shapley and Banzhaf values in the restricted game $\left(N, v^{\mathcal{F}}\right)$ by algorithm dividend requires a space $\Omega(|\mathcal{F}|)$ and a time $O(3^n)$.*

Proof. First, we calculate the dividends of the feasible coalitions with the *dividend* algorithm. It requires a time $O(3^n)$. The Shapley value for player $i \in N$ is given by formula (1). Observe that $|\{S \in \mathcal{F} : i \in S\}| < |\mathcal{F}|$, and hence the required time to evaluate the sum for every player is $O(|\mathcal{F}|)$. Thus, computing the Shapley value for n players in the restricted game requires a time $O(\max\{n|\mathcal{F}|, 3^n\})$. Also note that in the worst case $|\mathcal{F}| = 2^n$ and

$$\lim_{n\to\infty} \frac{n2^n}{3^n} = 0 \implies O(\max\{n|\mathcal{F}|, 3^n\}) = O(3^n).$$

The Banzhaf value for player $i \in N$ is given by formula (2). Since the time spent on $2^{|S \setminus i|}$ is $O(\log n)$ and the worst case is $|\mathcal{F}| = 2^n$, we have

$$\lim_{n\to\infty} \frac{n2^n \log n}{3^n} = 0 \implies O(\max\{n|\mathcal{F}|\log n, 3^n\}) = O(3^n).$$

The space complexity depends on the complexity of the dividends and so it is $\Omega(|\mathcal{F}|)$. □

Now we study the complexity for computing the Shapley value in the restricted game in two particular cases: the augmenting system of the connected subgraphs of a path P_n, and the augmenting system of the connected subgraphs of a star $K_{1,n-1}$.

5.1. The connected subgraphs of a path

Let $P_n = (1, 2, \ldots, n)$ be a path. The augmenting system $(N, \mathcal{F})$ of the connected subgraphs of P_n is given by

$$\mathcal{F} = \{[i,j] : 1 \le i \le j \le n\} \cup \{\emptyset\},$$

where $[i,j] = \{i, i+1, \ldots, j-1, j\}$. Notice that the non empty feasible coalitions can be arranged as entries on a upper-triangular matrix $A \in \mathbb{R}^{n\times n}$.

Example 7 *Let us consider the path P_5. Then the non empty coalitions of $\mathcal{F}$ are the entries of the following upper-triangular matrix*

$$\begin{pmatrix} 1 & 12 & 123 & 1234 & 12345 \\ & 2 & 23 & 234 & 2345 \\ & & 3 & 34 & 345 \\ & & & 4 & 45 \\ & & & & 5 \end{pmatrix}.$$

But then the number of feasible coalitions with i players is

$$T(i) = 5 - i + 1, \text{ for } 1 \leq i \leq 5.$$

In the same way, the number of coalitions which contain player k is given by $F_k = k(5 - k + 1)$, for $k \in \{1, \ldots, 5\}$.

Player	1	2	3	4	5
F_k	5	8	9	8	5

In order to study the complexity of computing the Shapley value in the restricted game we need the following results.

Lemma 1 *Let $(N, \mathcal{F})$ be the system of the connected subgraphs of a path P_n. Then it satisfies the following properties.*

1. *The number of coalitions which contain player k is $F_k = k(n - k + 1)$.*
2. *For $k \in \{1, \ldots, n\}$ we have $n \leq F_k \leq \frac{1}{4}(n+1)^2$.*
3. *The number of feasible coalitions with size i is $T(i) = n - i + 1$.*
4. *The total number of feasible coalitions is $|\mathcal{F}| = 1 + \frac{1}{2}n(n+1)$.*
5. *If $T \in \mathcal{F}$ with $|T| = i$, the number of non empty feasible coalitions contained in T is $C(i) = \frac{1}{2}i(i+1)$.*

Proof. 1. The non empty feasible coalitions can be arranged as entries on a upper-triangular matrix $A \in \mathbb{R}^{n \times n}$ such that the unitary coalitions are kept on the diagonal. On the row $k \in \{1, \ldots, n\}$ the coalitions of the set $\{[k, j] : k \leq j \leq n\}$ are arranged according to the path's order. Since the player k belongs to all coalitions of the submatrix (a_{ij}), for $1 \leq i \leq k$ and $k \leq j \leq n$, the total number of coalitions which contain player k is $F_k = k(n - k + 1)$.
2. The following expansion

$$\left(\frac{1}{2}(n+1)\right)^2 - \left(\frac{1}{2}(n+1) - k\right)^2 = (n + 1 - k)\, k = F_k,$$

implies that $F_k \leq \frac{1}{4}(n+1)^2$, for $k = 1, \ldots, n$. The minimum of F_k is obtained when $\left(\frac{1}{2}(n+1) - k\right)^2$ is maximum and it holds for $k = 1$ or $k = n$. That is to say, $\min F_k = F_1 = F_n = n$. On the other hand $\max F_k$ is reached for the central values

$$\begin{cases} k = p + 1 & \text{if } n = 2p + 1, \\ k = p, \ p + 1 & \text{if } n = 2p. \end{cases}$$

Then we obtain $F_{p+1} = \frac{1}{4}(n+1)^2$ if $n = 2p + 1$ and $F_p = F_{p+1} = \frac{1}{4}(n+1)^2 - \frac{1}{4}$ if $n = 2p$. Therefore $n \leq F_k \leq \frac{1}{4}(n+1)^2$, for $k = 1, \ldots, n$.
3. We observe that the entries on the i-th superdiagonal are the feasible coalitions with i players, where $i = 1, \ldots, n$. On this, there is a number of elements equal to $(n-i+1)$ and hence the number of feasible coalitions with i players is $T(i) = n-i+1$.

4. $|\mathcal{F}| = 1 + \sum_{i=1}^{n} T(i) = 1 + \sum_{i=1}^{n} (n - i + 1) = 1 + \frac{1}{2} n (n+1)$.
5. Let $T \in \mathcal{F}$ with $|T| = i$. Then T is a path with i vertices and property 4 implies that the number of non empty feasible coalitions contained in T is $C(i) = \frac{1}{2} i (i+1)$. □

Proposition 9 *Let $(N, \mathcal{F})$ be the system of the connected subgraphs of a path P_n and let (N, v) be a game. Computing the dividends in the restricted game $(N, v^{\mathcal{F}})$ requires a time $O(n^4)$ and a space $\Omega(n^2)$.*

Proof. We compute these dividends with the algorithm *dividend*. Then,

$$
\begin{aligned}
t(div) &= 1 + t(loop1) = 1 + \sum_{i=1}^{n} t(loop2) \\
&= 1 + \sum_{i=1}^{n} \sum_{j=1}^{T(i)} t(assignment) = 1 + \sum_{i=1}^{n} \sum_{j=1}^{T(i)} (1 + t(sum)) \\
&\leq 1 + \sum_{i=1}^{n} \sum_{j=1}^{T(i)} \left(1 + \sum_{k=1}^{C(i)} 2 \right) \\
&= 1 + \sum_{i=1}^{n} \left(1 + 2 \left(\frac{i^2 + i}{2} \right) \right) T(i) \\
&= 1 + \sum_{i=1}^{n} \left(i^2 + i + 1 \right) (n - i + 1) \\
&= 1 + n \sum_{i=1}^{n} \left(i^2 + i + 1 \right) - \sum_{i=1}^{n} i^3 + n \\
&= 1 + n \left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n \right) - \frac{n^2 (n+1)^2}{4} + n \\
&= \frac{n^4 + 6n^3 + 17n^2 + 12n + 12}{12}.
\end{aligned}
$$

Therefore, the time complexity is $O(n^4)$. On the other hand, if it is taken into account that $|\mathcal{F}| = 1 + \frac{1}{2} n (n+1)$ it is obtained that the space is $\Omega(n^2)$. □

Proposition 10 *Let $(N, \mathcal{F})$ be the system of the connected subgraphs of a path P_n and let (N, v) be a game. Computing the Shapley value in the restricted game $(N, v^{\mathcal{F}})$ by algorithm dividend requires a time $O(n^4)$ and a space $\Omega(n^2)$.*

Proof. First, we calculate all dividends of the feasible coalitions with the *dividend* algorithm. It requires a time $O(n^4)$. Next, we obtain the Shapley value for the player i with formula (1). Note that each player belongs to $F_i = i(n - i + 1)$ coalitions. Furthermore, $n \leq F_i \leq \frac{1}{4}(n+1)^2$ and hence the required time to evaluate the sum for every player is $O(n^2)$. Therefore, computing the Shapley value for n players requires a time $O(\max\{n^3, n^4\}) = O(n^4)$. The space complexity depends on the complexity of the dividends and so it is $\Omega(|\mathcal{F}|) = \Omega(n^2)$. □

Proposition 11 *Let $(N, \mathcal{F})$ be the system of the connected subgraphs of a path P_n and let (N, v) be a game. Computing the Banzhaf value in the restricted game $\left(N, v^{\mathcal{F}}\right)$ by algorithm dividend requires a time $O\left(n^4\right)$ and a space $\Omega(n^2)$.*

5.2. The connected subgraphs of a star

Let $K_{1,n-1}$ be a star on n vertices and let 1 be the center of star. The feasible coalitions of the augmenting system $(N, \mathcal{F})$ are the connected subgraphs of $K_{1,n-1}$, and it is given by $\mathcal{F} = \{S \subseteq N : 1 \in S \text{ or } |S| = 1\} \cup \{\emptyset\}$. Since 1 is the central vertex, we have

$$\mathcal{F} = \left\{\{1\} \cup S : S \in 2^{N \setminus \{1\}}\right\} \cup \{\{2\}, \ldots, \{n\}\} \cup \{\emptyset\}.$$

Lemma 2 *If $(N, \mathcal{F})$ is the system of the connected subgraphs of $K_{1,n-1}$, then it satisfies the following properties.*

1. *The number of coalitions which contain player k is*

$$F_k = \begin{cases} 2^{n-1} & \text{if } k = 1, \\ 2^{n-2} + 1 & \text{if } k \in ex(N). \end{cases}$$

2. *The number of feasible coalitions with $i \geq 2$ players is $T(i) = \binom{n-1}{i-1}$.*

3. *The total number of feasible coalitions is $|\mathcal{F}| = n + 2^{n-1}$.*

4. *If $T \in \mathcal{F}$ with $|T| = i$, then the number of non empty feasible coalitions contained in T is $C(i) = i + 2^{i-1} - 1$.*

Proof. 1. For $k = 1$, the set of connected subgraphs which contain 1 is $\left\{\{1\} \cup S : S \in 2^{N \setminus \{1\}}\right\}$ and hence $F_1 = 2^{n-1}$. If $k \in ex(N) = \{2, \ldots, n\}$, then S is a connected subgraph which contains k if and only if S is a connected subgraph which contains 1 in the star $K_{1,n-1} - \{k\}$ or $S = \{k\}$. Thus, $F_k = 2^{n-2} + 1$.
2. For $i \geq 2$, the number of coalitions with i players is the number of coalitions with $(i-1)$ players in the set $2^{N \setminus \{1\}}$. Therefore, $T(i) = \binom{n-1}{i-1}$.
3. The total number of feasible coalitions is

$$\begin{aligned} |\mathcal{F}| &= 1 + \sum_{i=1}^{n} T(i) = 1 + n + \sum_{i=2}^{n} \binom{n-1}{i-1} \\ &= n + 1 + \sum_{j=1}^{n-1} \binom{n-1}{j} = n + 2^{n-1}. \end{aligned}$$

4. Let T be a feasible coalition of size i. Then T is also a star with i vertices and property 3 implies that the number of non empty feasible coalitions contained in T is $C(i) = i + 2^{i-1} - 1$. □

Proposition 12 *Let* $(N, \mathcal{F})$ *be the system of the connected subgraphs of star* $K_{1,n-1}$ *and let* (N, v) *be a game. Computing the dividends in the restricted game* $\left(N, v^{\mathcal{F}}\right)$ *by algorithm dividend requires a time* $O\left(3^n\right)$ *and a space* $\Omega(2^n)$.

Proof. The algorithm *dividend* satisfies

$$\begin{aligned}
t(div) &= 1 + t(loop1) = 1 + \sum_{i=1}^{n} t\left(loop2\right) \\
&= 1 + \sum_{i=1}^{n} \sum_{j=1}^{T(i)} t(assignment) = 1 + \sum_{i=1}^{n} \sum_{j=1}^{T(i)} \left(1 + t(sum)\right) \\
&\leq 1 + \sum_{i=1}^{n} \sum_{j=1}^{T(i)} \left(1 + \sum_{k=1}^{C(i)} 2\right) = 1 + \sum_{i=1}^{n} \left(1 + 2\left(i + 2^{i-1} - 1\right)\right) T\left(i\right) \\
&= 1 + \sum_{i=1}^{n} \left(2i + 2^{i} - 1\right) \binom{n-1}{i-1} \\
&= 1 + 2\sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} + \sum_{j=0}^{n-1} 2^{j+1} \binom{n-1}{j} - \sum_{j=0}^{n-1} \binom{n-1}{j} \\
&= 1 + 2\sum_{j=0}^{n-1} j \binom{n-1}{j} + 2\sum_{j=0}^{n-1} 2^{j} \binom{n-1}{j} + \sum_{j=0}^{n-1} \binom{n-1}{j} \\
&= 1 + 2^{n-1}(n-1) + 3^{n-1}2 + 2^{n-1} = 1 + 2^{n-1}n + 3^{n-1}2.
\end{aligned}$$

Therefore, the time complexity is $O\left(3^n\right)$. On the other hand, if it is taken into account that $|\mathcal{F}| = n + 2^{n-1}$ is obtained that the space complexity is $\Omega\left(2^n\right)$. □

Proposition 13 *Let* $(N, \mathcal{F})$ *be the system of the connected subgraphs of star* $K_{1,n-1}$ *and let* (N, v) *be a game. Computing the Shapley value in the restricted game* $\left(N, v^{\mathcal{F}}\right)$ *by algorithm dividend requires a time* $O\left(3^n\right)$ *and a space* $\Omega\left(|\mathcal{F}|\right)$.

Proof. First, we calculate all dividends of the feasible coalitions with the *dividend* algorithm. It requires a time $O\left(3^n\right)$. The Shapley value for player $i \in N$ is given by formula (1). We know that each player belongs to F_k coalitions where

$$F_k = \begin{cases} 2^{n-1} & \text{if } k = 1, \\ 2^{n-2} + 1 & \text{if } k \in ex(N). \end{cases}$$

Evaluating the sum for every player requires $O\left(n2^n\right)$. Thus, computing the Shapley value for n players in the restricted game requires a time $O\left(\max\left\{n2^n, 3^n\right\}\right) = O\left(3^n\right)$. The space complexity is determined by the calculation of the dividends and hence it is $\Omega\left(|\mathcal{F}|\right)$. □

Proposition 14 *Let* $(N, \mathcal{F})$ *be the system of the connected subgraphs of star* $K_{1,n-1}$ *and let* (N, v) *be a game. Computing the Banzhaf value in the restricted game* $\left(N, v^{\mathcal{F}}\right)$ *by algorithm dividend requires a time* $O\left(3^n\right)$ *and a space* $\Omega\left(|\mathcal{F}|\right)$.

6. The complexity of the explicit formulas

Bilbao [2] showed *explicit formulas* (see Theorems 1, 2 and 3) in terms of v, for the Shapley and Banzhaf values of the players in the restricted game $\left(N, v^{\mathcal{F}}\right)$. Using these formulas is not necessary to compute the dividends of the restricted game. Now, we study the complexity of the explicit formulas.

Proposition 15 *Let $(N,\mathcal{F})$ be an augmenting system and let (N,v) be a game. We can compute the Shapley value in the restricted game $\left(N, v^{\mathcal{F}}\right)$, in a time $O\left(n\left|\mathcal{F}\right|\right)$ for every player.*

Proof. The Shapley value for player i, satisfies the formula

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{T\in\mathcal{F}\,:\,i\in T\}} \frac{(t-1)!\,t^*!}{t^+!} v(T) - \sum_{\{T\in\mathcal{F}\,:\,i\in T^*\}} \frac{t!\,(t^*-1)!}{t^+!} v(T). \qquad (3)$$

We have that each player belongs to F_i coalitions and $F_i < |\mathcal{F}|$. On the other hand $|\{T\in\mathcal{F} : i \in T^*\}| < |\mathcal{F}|$. The required time to evaluate the sums for every player is $O\left(n\left|\mathcal{F}\right|\right)$, because the factorials are evaluated in time $O\left(n\right)$. Thus, we can compute the Shapley value for player i in a time $O\left(n\left|\mathcal{F}\right|\right)$. □

Proposition 16 *Let $(N,\mathcal{F})$ be an augmenting system and let (N,v) be a game. We can compute the Banzhaf value in the restricted game $\left(N, v^{\mathcal{F}}\right)$, in a time $O\left(\log n\left|\mathcal{F}\right|\right)$ for every player.*

Proof. The Banzhaf value for the player i, satisfies the formula

$$\beta_i'\left(N, v^{\mathcal{F}}\right) = \sum_{\{T\in\mathcal{F}\,:\,i\in T\}} \frac{1}{2^{t^+-1}} v(T) - \sum_{\{T\in\mathcal{F}\,:\,i\in T^*\}} \frac{1}{2^{t^+-1}} v(T). \qquad (4)$$

Since $F_i < |\mathcal{F}|$ and $|\{T\in\mathcal{F} : i \in T^*\}| < |\mathcal{F}|$, the required time to evaluate the sums for every player is $O\left(\log n\left|\mathcal{F}\right|\right)$, considering that the powers of 2 are evaluated in time $O\left(\log n\right)$. Thus, we can compute the Shapley value for player i in a time $O\left(\log n\left|\mathcal{F}\right|\right)$. □

Proposition 17 *Let $(N,\mathcal{F})$ be an augmenting system and let (N,v) be a game such that $v(i) = 0$ for all $i \in N$. If $i \in ex(S)$ for all $S \in \mathcal{F}$ which contains i, we can compute the Shapley value for a player $i \in ex(S)$, in the restricted game $\left(N, v^{\mathcal{F}}\right)$ in a time $O\left(n\left|\mathcal{F}\right|\right)$.*

Proof. The Shapley value for the player $i \in \text{ex}\left(S\right)$, satisfies the formula

$$\Phi_i\left(N, v^{\mathcal{F}}\right) = \sum_{\{S\in\mathcal{F}\,:\,i\in S,\ |S|>1\}} \frac{(s-1)!\,s^*!}{s^+!} \left[v\left(S\right) - v(S\setminus i)\right].$$

Since $F_i < |\mathcal{F}|$ and the factorials are evaluated in time $O\left(n\right)$, the time to evaluate the sum for player i is $O\left(n\left|\mathcal{F}\right|\right)$. □

Proposition 18 *Let $(N,\mathcal{F})$ be an augmenting system and let (N,v) be a game such that $v(i) = 0$ for all $i \in N$. If $i \in ex(S)$ for all $S \in \mathcal{F}$ which contains i, we can compute the Banzhaf value for a player $i \in ex(S)$, in the restricted game $\left(N, v^{\mathcal{F}}\right)$ in a time $O\left(\log n\left|\mathcal{F}\right|\right)$.*

Proof. The Banzhaf value for the player $i \in \mathrm{ex}(S)$, satisfies the formula

$$\beta_i'\left(N, v^{\mathcal{F}}\right) = \sum_{\{S \in \mathcal{F} : i \in S,\ |S|>1\}} \frac{1}{2^{s^+-1}} \left[v(S) - v(S \setminus i)\right].$$

Since $F_i < |\mathcal{F}|$ and the powers of 2 are evaluated in time $O(\log n)$, the time to evaluate the sum for player i is $O(\log n |\mathcal{F}|)$. □

Let $(N, \mathcal{F})$ be the system of the connected subgraphs of a path P_n and let (N, v) be a game. We can compute the Shapley and Banzhaf value in the restricted game $\left(N, v^{\mathcal{F}}\right)$ using Theorem 1 without computing the dividends. To study the complexity we need the following lemma.

Lemma 3 *Let $(N, \mathcal{F})$ be the system of the connected subgraphs of a path P_n. For every player i, the cardinality $|\{T \in \mathcal{F} : i \in T^*\}| = n$.*

Proof. For every player i, the coalitions of the set $\mathcal{F}_i^* = \{T \in \mathcal{F} : i \in T^*\}$ can be arranged in the n entries of the row i on a matrix B. We can obtain the matrix B with the following procedure:

1. $B = A$, where A is the matrix of the non empty feasible coalitions.

2. For $2 \leq i \leq n$ and $j < i$, do $b_{ij} = b_{j,i-1}$.

3. For every row $i \in \{1, \ldots, n\}$ the player i is removed of all the coalitions of this row.

4. Finally, the empty coalition is stored in the entries corresponding to the main diagonal.

□

Example 8 *The augmenting system of the connected subgraphs of a path P_5 is $\mathcal{F} = \{[i, j] : 1 \leq i \leq j \leq 5\} \cup \{\emptyset\}$ and $ex(N) = \{1, 5\}$.*
1. $B = A$, being A the matrix that stores the non empty feasible coalitions.

$$B = \begin{pmatrix} 1 & 12 & 123 & 1234 & 12345 \\ & 2 & 23 & 234 & 2345 \\ & & 3 & 34 & 345 \\ & & & 4 & 45 \\ & & & & 5 \end{pmatrix}$$

2. For $2 \leq i \leq 5$ and $j < i$, do $b_{ij} = b_{j,i-1}$.

$$B = \begin{pmatrix} 1 & 12 & 123 & 1234 & 12345 \\ 1 & 2 & 23 & 234 & 2345 \\ 12 & 2 & 3 & 34 & 345 \\ 123 & 23 & 3 & 4 & 45 \\ 1234 & 234 & 34 & 4 & 5 \end{pmatrix}$$

3. For every row $i \in \{1, \ldots, 5\}$ the player i is removed of all the coalitions of this row.

$$B = \begin{pmatrix} & 2 & 23 & 234 & 2345 \\ 1 & & 3 & 34 & 345 \\ 12 & 2 & & 4 & 45 \\ 123 & 23 & 3 & & 5 \\ 1234 & 234 & 34 & 4 & \end{pmatrix}$$

4. Finally, the empty coalition is stored in the entries corresponding to the main diagonal.

$$B = \begin{pmatrix} \emptyset & 2 & 23 & 234 & 2345 \\ 1 & \emptyset & 3 & 34 & 345 \\ 12 & 2 & \emptyset & 4 & 45 \\ 123 & 23 & 3 & \emptyset & 5 \\ 1234 & 234 & 34 & 4 & \emptyset \end{pmatrix} \begin{matrix} \mathcal{F}_1^* \\ \mathcal{F}_2^* \\ \mathcal{F}_3^* \\ \mathcal{F}_4^* \\ \mathcal{F}_5^* \end{matrix}$$

where $\mathcal{F}_i^* = \{T \in \mathcal{F} : i \in T^*\}$

Example 9 *The augmenting system of the connected subgraphs of a path P_5 is $\mathcal{F} = \{[i,j] : 1 \leq i \leq j \leq 5\} \cup \{\emptyset\}$ and $ex(N) = \{1,5\}$. For a feasible coalition $S \in \mathcal{F}$, we consider the set $S^* = \{i \in N \setminus S : S \cup i \in \mathcal{F}\}$ of augmentations of S and the set $S^+ = S \cup S^* = \{i \in N : S \cup i \in \mathcal{F}\}$.*

S	S^*	S^+
$\emptyset$	12345	12345
1	2	12
2	13	123
3	24	234
4	35	345
5	4	45
12	3	123
23	14	1234
34	25	2345
45	3	345
123	4	1234
234	15	12345
345	2	2345
1234	5	12345
2345	1	12345
12345	$\emptyset$	12345

Proposition 19 *Let $(N, \mathcal{F})$ be the system of the connected subgraphs of a path P_n, and let (N, v) be a game. Computing the Shapley value, for every player, in the restricted game $\left(N, v^{\mathcal{F}}\right)$ requires a time $O\left(n^3\right)$ and a space $\Omega(n^2)$.*

Proof. The Shapley value for player $i \in N$ is given by formula (3). Since each player belongs to F_i coalitions with $n \leq F_i \leq \frac{1}{4}(n+1)^2$, for $i = 1, \ldots, n$ and $|\mathcal{F}_i^*| = n$, the time is $O\left(\max\left\{n^3, n^2\right\}\right) = O\left(n^3\right)$. The space complexity is determined by the storage of the function v for the feasible coalitions, and $|\mathcal{F}| = 1 + \frac{1}{2}n\left(n+1\right)$. Therefore, the space is $\Omega(n^2)$. □

Proposition 20 *Let $(N, \mathcal{F})$ be the system of the connected subgraphs of a path P_n,and let (N, v) be a game. Computing the Banzhaf value in the restricted game $\left(N, v^{\mathcal{F}}\right)$ requires a time $O\left(n^2 \log n\right)$ and a space $\Omega(n^2)$.*

Proof. The Banzhaf value for player $i \in N$ is given by formula (4). Since F_i satisfies $n \leq F_i \leq \frac{1}{4}(n+1)^2$, for $i = 1, \ldots, n$ and $|\mathcal{F}_i^*| = n$, the time complexity is $O\left(\max\left\{n^2 \log n, n \log n\right\}\right) = O\left(n^2 \log n\right)$. □

Notice that if player $i \in ex(N)$, then $F_i = n$. In this case, we obtain the following results.

Corollary 1 *Let $(N, \mathcal{F})$ be the system of the connected subgraphs of a path P_n,and let (N, v) be a game. If $i \in ex(N) = \{1, n\}$ then*

1. *Computing $\Phi_i\left(N, v^{\mathcal{F}}\right)$ requires a time $O\left(n^2\right)$.*
2. *Computing $\beta_i'\left(N, v^{\mathcal{F}}\right)$ requires a time $O\left(n \log n\right)$.*

Let $(N, \mathcal{F})$ be the augmenting system of the connected subgraphs of star $K_{1,n-1}$ and (N, v) a game. If (N, v) is a game such that $v(i) = 0$ for all $i \in N$, we can compute the Shapley value in the restricted game $\left(N, v^{\mathcal{F}}\right)$ using Theorem 3 without computing the dividends.

Proposition 21 *Let $(N, \mathcal{F})$ be the system of the connected subgraphs of star $K_{1,n-1}$ and let (N, v) be a game. If (N, v) is a game such that $v(i) = 0$ for all $i \in N$, computing the Shapley value in the restricted game $\left(N, v^{\mathcal{F}}\right)$ requires a time $O\left(n2^n\right)$ and a space $\Omega\left(|2^n|\right)$.*

Proof. Using Theorem 3 we have that

$$\begin{aligned}
\Phi_1\left(N, v^{\mathcal{F}}\right) &= \sum_{\{S \in \mathcal{F}\,:\, 1 \in S,\ |S|>1\}} \frac{(s-1)!(n-s)!}{n!} v(S), \\
\Phi_i\left(N, v^{\mathcal{F}}\right) &= \sum_{\{S \in \mathcal{F}\,:\, i \in S,\ |S|>1\}} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v(S \setminus i)\right],
\end{aligned}$$

for $i \in \{2, \ldots, n\}$. We have that each player belongs to F_i coalitions where

$$F_i = \begin{cases} 2^{n-1} & \text{if } i = 1, \\ 2^{n-2} + 1 & \text{if } i \in ex(N). \end{cases}$$

The required time to evaluate both Φ_1 and Φ_i for every player is $O\left(nF_i\right)$, because the factorials are evaluated in time $O\left(n\right)$. Thus, we can compute the Shapley value for player i in a time $O\left(nF_i\right) = O\left(n2^n\right)$. The space complexity is determined by the storage of the function v for the feasible coalitions, and $|\mathcal{F}| = n + 2^{n-1}$. Therefore, the space is $\Omega(2^n)$. □

Acknowledgements

We would like to express our gratitude to an anonymous referee for interesting comments and for their careful reading of the manuscript.

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