

A DUAL APPROACH TO COMPROMISE VALUES

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ABSTRACT. The aim of this paper is to analyse the *Tijs value* as a solution concept for cost allocation problems. Given a cost game c , we define its *dual τ -value* as $\tau(c^*)$, if the dual game c^* is quasi-balanced. Then we show that the dual τ -value coincides, for a wide class of cost games, with the *alternate cost avoided (ACA)* allocation proposed in the 1930's by the Tennessee Valley Authority. It turns out that the center of the imputation set, the egalitarian nonseparable contribution and the *ACA* allocation are collinear.

1. INTRODUCTION

A transferable utility game is a pair (N, v) , where N is a finite set and $v : 2^N \rightarrow \mathbb{R}$, is a function with $v(\emptyset) = 0$. The elements of $N = \{1, \dots, n\}$ are called *players*, the subsets $S \in 2^N$ *coalitions* and $v(S)$ is the *worth* of S . The worth of a coalition is interpreted as the maximal profit or minimal cost for the players in their own coalition. We will consider *profit games* if $v(S)$ measures the profit of the coalition S and *cost games* $c : 2^N \rightarrow \mathbb{R}$ if the function measures the cost $c(S)$ incurred by S . We will use a shorthand notation and write i for the set $\{i\}$, $S \cup i$ for $S \cup \{i\}$, and $S \setminus i$ for $S \setminus \{i\}$.

In a game (N, v) , a vector $x \in \mathbb{R}^n$ is called *efficient* if it distributes the worth $v(N)$ among the players, i.e., $\sum_{i \in N} x_i = v(N)$. The set of all efficient vectors is called the *preimputation set* and is denoted by $I^*(v)$. The *imputation set* of a profit game (N, v) is defined by

$$I(v) = \{x \in I^*(v) : x_i \geq v(i) \text{ for all } i \in N\}.$$

Note that $I(v) \neq \emptyset$ if and only if $v(N) \geq \sum_{i \in N} v(i)$. If (N, c) is a cost game, then the set of (efficient) vectors $x \in \mathbb{R}^n$ with $\sum_{i \in N} x_i = c(N)$ is denoted by $I^*(c)$, and its imputation set is

$$I(c) = \{x \in I^*(c) : x_i \leq c(i) \text{ for all } i \in N\}.$$

Therefore, $I(c) \neq \emptyset$ if and only if $c(N) \leq \sum_{i \in N} c(i)$. Assuming that the coalition N of all players will be formed, the following solution concept

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will prescribe a distribution of the cost $c(N)$ or the profit $v(N)$ among the players.

Definition 1.1. *The core of a cost game c and the core of a profit game v are respectively defined by*

$$\begin{aligned} \text{Core}(c) &= \{x \in I^*(c) : x(S) \leq c(S) \text{ for all } S \subseteq N\}, \\ \text{Core}(v) &= \{x \in I^*(v) : x(S) \geq v(S) \text{ for all } S \subseteq N\}, \end{aligned}$$

where $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$.

Games with a nonempty core are called *balanced games*.

Definition 1.2. *The dual game (N, v^*) of (N, v) is defined by the dual function $v^* : 2^N \rightarrow \mathbb{R}$, where $v^*(S) = v(N) - v(N \setminus S)$, for all $S \subseteq N$.*

Note that $(v^*)^* = v$. We remark that if (N, v) is a profit (cost) game and (N, v^*) is a cost (profit) game, then $\text{Core}(v) = \text{Core}(v^*)$.

The *Tijs value* of a game is a feasible compromise between the upper and the lower vectors for the game, introduced by Tijs [4, 6, 7]. The *upper vector* of the game (N, v) is the vector $M(v) \in \mathbb{R}^n$, where

$$M_i(v) = v(N) - v(N \setminus i),$$

for all $i \in N$. The component $M_i(v)$ is called the utopia payoff for player i in the grand coalition. The remainder of $i \in S$ if the coalition S forms and all other players in S obtain their utopia payoff is

$$R^v(S, i) = v(S) - \sum_{j \in S \setminus i} M_j(v).$$

The *lower vector* is the vector $m(v) \in \mathbb{R}^n$, defined by

$$m_i(v) = \max_{\{S \subseteq N : i \in S\}} R^v(S, i).$$

Definition 1.3. *A game (N, v) is called quasi-balanced if it satisfies:*

1. $m(v) \leq M(v)$,
2. $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$.

The family of quasi-balanced games contains the family of balanced games as a subset (see Tijs [4]). For a quasi-balanced game v the Tijs value, denoted by $\tau(v)$, is the unique preimputation (efficient vector) on the closed interval $[m(v), M(v)]$ in \mathbb{R}^n . So we have

$$\tau(v) = m(v) + \lambda(M(v) - m(v)),$$

where $\lambda \in \mathbb{R}$ is determined by

$$\sum_{i \in N} \tau_i(v) = v(N).$$

2. COST ALLOCATIONS PROBLEMS

The Tennessee Valley Authority was a development project to construct dams and reservoirs along the Tennessee River. A method used by civil engineers to allocate the costs of the project is known as the *alternate cost avoided* (ACA) allocation.

We now describe the version of this method given by Young [8]. Let (N, c) be a cost game. For each project $i \in N$ we define its *separable cost*

$$s_i = c(N) - c(N \setminus i).$$

The *alternate cost* for i is the cost $c(i)$ and the difference

$$r_i = c(i) - s_i$$

is the *alternate cost avoided*. Let $s(N) = \sum_{i \in N} s_i$ and $r(N) = \sum_{i \in N} r_i$. Then the ACA allocation assigns to each $i \in N$ the cost allocation given by the formula

$$ACA_i(c) = \begin{cases} s_i + \frac{r_i}{r(N)}(c(N) - s(N)), & \text{if } r(N) \neq 0, \\ s_i, & \text{otherwise.} \end{cases} \quad (2.1)$$

Notice that every i pays the sum of its separable cost and a proportion of the *nonseparable cost* $c(N) - s(N)$. Notice further that $r_i \geq 0$ for all $i \in N$ if c is *subadditive*, i.e., for all $S, T \in 2^N$ such that $S \cap T = \emptyset$, we have $c(S \cup T) \leq c(S) + c(T)$.

Definition 2.1. Let (N, c) be a cost game. The associated savings game (N, v) is defined by $v(S) = \sum_{i \in S} c(i) - c(S)$, for all $S \subseteq N$.

The worth $v(S)$ is the cost savings obtained from cooperation between the players. Driessen [2, Chapter IV] obtained the relationship between two separable cost allocation methods and the Tijs value of the associated savings game.

If the cost game c is subadditive, then for every $S \subset N$ and $i \in N \setminus S$, we have that $c(S \cup i) \leq c(S) + c(i)$. It follows that

$$v(S \cup i) - v(S) = c(i) - (c(S \cup i) - c(S)) \geq 0.$$

We conclude that its savings game is monotonic and hence nonnegative.

Let (N, c) be a cost game. Then its savings game (N, v) is monotonic if and only if for all $T \subseteq S \subseteq N$,

$$\sum_{i \in T} c(i) - c(T) \leq \sum_{i \in S} c(i) - c(S) \iff c(S) \leq c(T) + \sum_{i \in S \setminus T} c(i). \quad (2.2)$$

The monotonicity of the savings game implies, for all $i \in N$,

$$s_i = c(N) - c(N \setminus i) \leq c(i), \text{ or } r_i \geq 0.$$

Tijs and Driessen [5] introduced the *reverse τ -value*, $\tau^r(c)$, of a cost game (N, c) defined as $\tau^r(c) := -\tau(-c)$ if $-c$ is quasi-balanced. They proved that the reverse τ -value of a cost game c coincides with the ACA allocation if the cost game is such that $m_i^r(c) = c(i)$ for all $i \in N$, i.e., if $-c$ is semi-convex.

3. THE DUAL TIJS VALUE

Given a cost game c , we define its *dual τ -value* as $\tau^*(c) := \tau(c^*)$, if the dual game c^* is quasi-balanced. First, we study the properties of the upper and the lower vectors for the game (N, c^*) .

Theorem 3.1. *Let (N, c) be a cost game such that its savings game is monotonic. Then the components of the upper and the lower vectors of c^* satisfy*

1. $M_i(c^*) = c(i)$ for all $i \in N$.
2. $m_i(c^*) = c(N) - c(N \setminus i)$ for all $i \in N$.
3. $m_i(c^*) \leq M_i(c^*)$ for all $i \in N$, and $c^*(N) \leq \sum_{i \in N} M_i(c^*)$.

Proof. 1. For every $i \in N$,

$$M_i(c^*) = c^*(N) - c^*(N \setminus i) = c(N) - (c(N) - c(i)) = c(i).$$

2. For every $i \in N$,

$$\begin{aligned} m_i(c^*) &= \max_{\{S \subseteq N : i \in S\}} \left\{ c^*(S) - \sum_{j \in S \setminus i} M_j(c^*) \right\} \\ &= \max_{\{S \subseteq N : i \in S\}} \left\{ c(N) - c(N \setminus S) - \sum_{j \in S \setminus i} c(j) \right\} \\ &= c(N) - \min_{\{S \subseteq N : i \in S\}} \left\{ c(N \setminus S) + \sum_{j \in S \setminus i} c(j) \right\} \\ &= c(N) - \min_{T \subseteq N \setminus i} \left\{ c(T) + \sum_{j \in (N \setminus i) \setminus T} c(j) \right\}. \end{aligned}$$

From the monotonicity of the associated savings game and (2.2) we obtain

$$c(N \setminus i) \leq c(T) + \sum_{j \in (N \setminus i) \setminus T} c(j),$$

for all $T \subseteq N \setminus i$. Therefore,

$$\min_{T \subseteq N \setminus i} \left\{ c(T) + \sum_{j \in (N \setminus i) \setminus T} c(j) \right\} = c(N \setminus i),$$

and we conclude that $m_i(c^*) = c(N) - c(N \setminus i)$.

3. The inequality (2.2) implies that $c(N) \leq c(N \setminus i) + c(i)$. From 1 and 2, it follows that $m_i(c^*) = c(N) - c(N \setminus i) \leq c(i) = M_i(c^*)$. Moreover,

$$c^*(N) = c(N) \leq \sum_{i \in N} c(i) = \sum_{i \in N} M_i(c^*). \quad \square$$

Theorem 3.2. *Let (N, c) be a cost game such that its savings game is monotonic. Then*

1. *The dual game c^* is quasi-balanced if and only if $s(N) \leq c(N)$.*
2. *If $s(N) \leq c(N)$, then the dual Tijs value $\tau(c^*)$ coincides with the allocation $ACA(c)$.*

Proof. 1. It follows from Theorem 3.1.3 that the dual game c^* is quasi-balanced if and only if

$$\sum_{i \in N} (c(N) - c(N \setminus i)) \leq c(N), \quad \text{or} \quad s(N) \leq c(N).$$

2. For every $i \in N$, the component i of the dual Tijs value is

$$\tau_i(c^*) = s_i + \lambda(c(i) - s_i) = s_i + \lambda r_i,$$

where $\lambda \in \mathbb{R}$ is such that

$$\sum_{i \in N} \tau_i(c^*) = c(N).$$

Since $r_i \geq 0$ we have

$$r(N) = 0 \iff r_i = 0 \text{ for all } i \in N \iff \tau(c^*) = M(c^*) = m(c^*).$$

Otherwise,

$$c(N) = \sum_{i \in N} (s_i + \lambda r_i) = s(N) + \lambda r(N)$$

implies that

$$\lambda = \frac{c(N) - s(N)}{r(N)}.$$

Thus, the components of the dual Tijs value satisfy

$$\tau_i(c^*) = \begin{cases} s_i, & \text{if } r(N) = 0, \\ s_i + \frac{r_i}{r(N)} (c(N) - s(N)), & \text{if } r(N) > 0, \end{cases}$$

which coincides with the formula (2.1) of the *ACA* allocation. \square

4. COLLINEARITY OF COST ALLOCATION RULES

The center of the imputation set (*CIS*) and the egalitarian nonseparable contribution (*ENSC*) are well known cost allocation rules (cf. [1], [2], [3]).

Definition 4.1. *Let (N, c) be a cost game. For each $i \in N$, the *CIS* and the *ENSC* allocations are given by*

$$\begin{aligned} CIS_i(c) &= c(i) + \frac{1}{n} \left(c(N) - \sum_{i \in N} c(i) \right), \\ ENSC_i(c) &= s_i + \frac{1}{n} (c(N) - s(N)). \end{aligned}$$

Obviously $CIS(c) = ENSC(c^*)$ and $ENSC(c) = CIS(c^*)$. In the next theorem we compute the reverse τ -value of the dual of a cost game (N, c) , that is, the vector $\tau^r(c^*) = -\tau(-c^*)$. This vector is the unique efficient compromise between the two vectors $M^r(c^*)$ and $m^r(c^*)$.

Theorem 4.1. *Let (N, c) be a cost game such that its savings game (N, v) is nonnegative. Then we have*

1. $M_i^r(c^*) = c(i)$ for all $i \in N$.
2. $m_i^r(c^*) = c(i) + \left(c(N) - \sum_{j \in N} c(j)\right)$ for all $i \in N$.
3. The reverse τ -value of the dual game c^* coincides with the allocation $CIS(c)$.

Proof. 1. $M_i^r(c^*) = -M_i(-c^*) = c^*(N) - c^*(N \setminus i) = c(i)$, for all $i \in N$.

2. The components of the lower vector satisfy

$$\begin{aligned} m_i^r(c^*) &= -m_i(-c^*) \\ &= - \max_{\{S \subseteq N : i \in S\}} \left\{ -c^*(S) - \sum_{j \in S \setminus i} M_j(-c^*) \right\} \\ &= - \max_{\{S \subseteq N : i \in S\}} \left\{ c(N \setminus S) - c(N) + \sum_{j \in S \setminus i} c(j) \right\} \\ &= c(N) - \max_{T \subseteq N \setminus i} \left\{ c(T) + \sum_{j \in (N \setminus i) \setminus T} c(j) \right\}. \end{aligned}$$

For every $T \subseteq N \setminus i$ we have that

$$\begin{aligned} c(T) + \sum_{j \in (N \setminus i) \setminus T} c(j) &= c(T) + \sum_{j \in N \setminus i} c(j) - \sum_{j \in T} c(j) \\ &= \sum_{j \in N \setminus i} c(j) - v(T) \\ &\leq \sum_{j \in N \setminus i} c(j), \end{aligned}$$

since the savings game (N, v) is nonnegative. Moreover, this upper bound is attained in $T = \emptyset$ and hence

$$m_i^r(c^*) = c(N) - \sum_{j \in N \setminus i} c(j) = c(i) + \left(c(N) - \sum_{j \in N} c(j) \right),$$

for all $i \in N$.

3. First, we show that the game $-c^*$ is quasi-balanced. By hypothesis $v(N) \geq 0$ and hence $m_i^r(c^*) = c(i) - v(N) \leq c(i) = M_i^r(c^*)$, for all $i \in N$.

Furthermore,

$$\sum_{i \in N} c(i) - c(N) \geq 0 \text{ implies } c(N) \leq \sum_{i \in N} M_i^r(c^*).$$

Finally,

$$\begin{aligned} \sum_{i \in N} m_i^r(c^*) &= \sum_{i \in N} c(i) + nc(N) - n \sum_{i \in N} c(i) \\ &= c(N) + (n-1) \left(c(N) - \sum_{i \in N} c(i) \right) \\ &= c(N) - (n-1)v(N) \leq c(N). \end{aligned}$$

Therefore, the reverse τ -value is defined and equals the unique efficient vector between the upper and lower vectors with components $M_i^r(c^*) = c(i)$ and $m_i^r(c^*) = c(i) - v(N)$. Then

$$\tau_i^r(c^*) = c(i) - v(N) + \lambda v(N) = c(i) + (\lambda - 1)v(N).$$

The efficiency implies that

$$c(N) = \sum_{i \in N} \tau_i^r(c^*) = \sum_{i \in N} c(i) + n(\lambda - 1)v(N).$$

From this we conclude that for all $i \in N$,

$$\tau_i^r(c^*) = c(i) + \frac{1}{n} \left(c(N) - \sum_{i \in N} c(i) \right),$$

which coincides with the *CIS* allocation. \square

In order to obtain the *collinearity* of the three allocations rules *CIS*, *ENSC* and *ACA*, it is useful to recall some properties of orthogonal projections of \mathbb{R}^n onto a hyperplane. The *inner product* of vectors x and y is the number $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. Then a hyperplane H in \mathbb{R}^n is a set of the form $H = \{x \in \mathbb{R}^n : \langle m, x \rangle = \alpha\}$, where m is a nonzero vector normal to H and α is a real number. The orthogonal projection $P_H : \mathbb{R}^n \rightarrow H$ onto H is given by

$$P_H(q) = q + \frac{\alpha - \langle m, q \rangle}{\|m\|^2} m,$$

for each $q \in \mathbb{R}^n$. Note that

(P1) $P_H(q) = q$ if and only if $q \in H$,

(P2) P_H is an affine map, i.e., $P_H(\lambda q + \eta r) = \lambda P_H(q) + \eta P_H(r)$ for all $q, r \in \mathbb{R}^n$ and $\lambda, \eta \in \mathbb{R}$ with $\lambda + \eta = 1$.

From (P2) it follows that the orthogonal projection on H maps a line in \mathbb{R}^n onto a line (or a point) in H . This implies:

(P3) If $p, q, r \in \mathbb{R}^n$ are collinear, i.e., there is a line in \mathbb{R}^n containing p, q and r , then also $P_H(p), P_H(q)$ and $P_H(r)$ are collinear. Furthermore, if $p = \lambda q + \eta r$ with $\lambda + \eta = 1$, then $P_H(p) = \lambda P_H(q) + \eta P_H(r)$.

In the next proposition we obtain the orthogonal projections of the vectors $k = (c(i))_{i \in N}$ and $s = (s_i)_{i \in N}$ on the preimputation set $I^*(c)$. For a geometric illustration see Figure 1.

Proposition 4.2. *Let H be the preimputation set $I^*(c)$ of a cost game (N, c) and let k, s be the vectors defined above. Then $P_H(k) = CIS(c)$ and $P_H(s) = ENSC(c)$.*

Proof. Note that $H = \{x \in \mathbb{R}^n : \langle m, x \rangle = c(N)\}$, where $m = (1, \dots, 1)$. Then

$$P_H(k) = k + \frac{1}{n} \left(c(N) - \sum_{i=1}^n c(i) \right) (1, \dots, 1),$$

which implies

$$(P_H(k))_i = c(i) + \frac{1}{n} \left(c(N) - \sum_{i=1}^n c(i) \right) = CIS_i(c),$$

for each $i \in N$. Further,

$$P_H(s) = s + \frac{1}{n} \left(c(N) - \sum_{i=1}^n s_i \right) (1, \dots, 1).$$

Hence, for each $i \in N$, we obtain

$$(P_H(s))_i = s_i + \frac{1}{n} (c(N) - s(N)) = ENSC_i(c). \quad \square$$

Theorem 4.3. *Let (N, c) be a cost game with $s(N) \leq c(N) \leq k(N)$. Then the allocations $CIS(c)$, $ENSC(c)$ and $ACA(c)$ are collinear. Moreover, if $s(N) \neq k(N)$ then*

$$ACA(c) = ENSC(c) + \frac{c(N) - s(N)}{r(N)} (CIS(c) - ENSC(c)).$$

Otherwise, $ACA(c) = ENSC(c)$.

Proof. The definition (2.1) of the ACA allocation implies that

$$ACA(c) = s + \frac{c(N) - s(N)}{k(N) - s(N)} (k - s)$$

if $s(N) \neq k(N)$, and $ACA(c) = s$, otherwise. Thus the vector $ACA(c)$ is a convex combination of the vectors $k = (c(i))_{i \in N}$ and $s = (s_i)_{i \in N}$. Further

$ACA(c) \in I^*(c) = H$. In the case that $s(N) \neq k(N)$ the properties (P1), (P2) and (P3) imply that

$$ACA(c) = P_H(ACA(c)) = P_H(s) + \frac{c(N) - s(N)}{k(N) - s(N)} (P_H(k) - P_H(s)).$$

Then, by Proposition 4.2, we obtain

$$ACA(c) = ENSC(c) + \frac{c(N) - s(N)}{r(N)} (CIS(c) - ENSC(c)).$$

If $s(N) = k(N)$ then $s(N) = c(N)$ and so $ACA(c) = ENSC(c)$. \square

Remark 4.1. *If $s(N) = k(N)$ and the savings game is monotonic, then the allocations $ACA(c) = CIS(c) = ENSC(c)$.*

Theorem 4.3 suggests us to define the following concept.

Definition 4.2. *The compromise set of a cost game (N, c) is the convex hull of the vectors $CIS(c)$ and $ENSC(c)$.*

In future research, we expect to see new applications of this set, consisting of the line segment with end points $CIS(c)$ and $ENSC(c)$.

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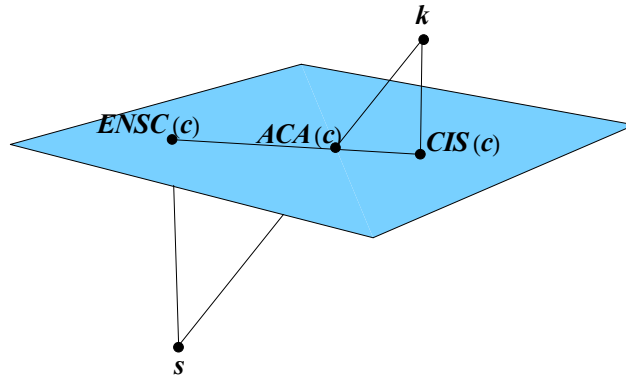


Figure 1. Orthogonal projections on $I^*(c)$