The τ-value for games on matroids

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Abstract

In the classical model of games with transferable utility one assumes that each subgroup of players can form and cooperate to obtain its value. However, we can think that in some situations this assumption is not realistic, that is, not all coalitions are feasible. This suggests that it is necessary to raise the whole question of generalizing the concept of transferable utility game, and therefore to introduce new solution concepts. In this paper we define games on matroids and extend the τ-value as a compromise value for these games.

Keywords: τ-value, matroid, cooperative game

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1 Introduction

Faigle and Kern [5] proposed a model in which cooperation among players is restricted to a certain family of subsets of players, the *feasible* coalitions of the game. Their idea is to restrict the allowable coalitions by using underlying partially ordered sets and they obtained an extension of the Shapley value for games on distributive lattices. Their model is a special case of the games on convex geometries studied by Bilbao and Edelman [1]. In the present paper, we will define the feasible coalitions by using combinatorial geometries called *matroids*. Games on matroids are introduced by Bilbao, Driessen, Jiménez-Losada and Lebrón [2]. Let us assume that there are two rules of cooperation between players:

- If a coalition can form, then any subcoalition is also feasible because if the players that take part in the formation of a coalition have common interests, a subset of these players has at least the same common interests.
- If there are two coalitions where the cardinality differs one, there is a player of the largest coalition which can join the smallest making a feasible coalition.

The characteristic function of a game on a matroid is only defined on the elements of the matroid. We propose a generalization of the τ-value introduced by Tijs [6, 8] as a concept of compromise value for games defined on matroids. Let us briefly outline the content. In the next section, we define matroids, describe some of their properties and introduce games on matroids. In the third section we extend the concept of the τ-value to games on matroids and obtain a family of axioms that characterize uniquely this value. Finally, we build two special classes of games on matroids in the last section which have a nice expression of the τ-value. Now, we introduce several concepts from the theory of cooperative games.

Definition 1.1 *A transferable utility game is a pair* (N, v)*, where* N *is a finite set and* $v : 2^N \to \mathbb{R}$*, is a function with* $v(\emptyset) = 0$*.*

The elements of $N = \{1, \ldots, n\}$ are called *players*, the subsets $S \in 2^N$ *coalitions* and $v(S)$ is the *worth* of S. The worth of a coalition is interpreted as the maximal profit or minimal cost for the players in that coalition. The function v is called the *characteristic function*. Given a game (N, v) and a coalition $S \subseteq N$, the *subgame* (S, v) is obtained by restricting v to 2^S. By Γ^N we denote the set of all games (N, v) where N is the set of players. We will use a shorthand notation and write $S \cup i$ for $S \cup \{i\}$, and $S \setminus i$ for $S \setminus \{i\}$.

In a game (N, v), a vector $x \in \mathbb{R}^n$ is called *efficient* if it distributes the worth $v(N)$ among the players, i.e., $\sum_{i \in N} x_i = v(N)$. The set of all efficient vectors is called the *preimputation set* and is denoted by $I^*(v)$. The *imputation set* is defined by

$$I(v) := \{x \in I^*(v) : x_i \geq v(i) \text{ for all } i \in N\}.$$

Note that $I(v) \neq \emptyset$ if and only if $v(N) \geq \sum_{i \in N} v(i)$. Assuming that the coalition N of all players will be formed, a one-point solution concept will prescribe a distribution of the worth $v(N)$ among the players. Given a vector $x \in \mathbb{R}^n$ and a coalition S, we define $x(S) := \sum_{i \in S} x_i$ and $x(\emptyset) = 0$.

Definition 1.2 *The core of a game* (N, v) *is the set*

$$C(v) := \{x \in I^*(v) : x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

Games with a nonempty core are called balanced games.

The τ*-value* of a game is a compromise between the upper and the lower vectors for the game introduced by Tijs [6] (see the survey [8]). The *upper vector* of the game (N, v) is the n-vector M^v, where $M_i^v := v(N) - v(N \setminus i)$, for all $i \in N$. The component M_i^v is called utopia payoff for player i in the grand coalition. The remainder of $i \in S$ if the coalition S forms and all other players in S obtain their utopia payoff is $R^v(S, i) := v(S) - M^v(S \setminus i)$. The *lower vector* is the vector $m^v \in \mathbb{R}^n$, defined by

$$m_i^v = \max_{\{S : i \in S\}} R^v(S, i).$$

Definition 1.3 *A game* (N, v) *is called quasi-balanced if it satisfies:*

(QB1) $m^v \leq M^v$, *and*

(QB2) $m^v(N) \leq v(N) \leq M^v(N)$.

The family of quasi-balanced games is a full-dimensional cone in the $(2^n - 1)$-dimensional vector space Γ^N and contains the family of balanced games as a subset [6]. For a quasi-balanced game v the τ-value, denoted by τ^v, is the unique preimputation (efficient vector) on the closed interval $[m^v, M^v]$ in $\mathbb{R}^n$. Then we have $\tau^v = m^v + \lambda (M^v - m^v)$, where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \tau_i^v = v(N)$.

2 Matroids and games

Let $N = \{1, \ldots, n\}$ be a finite set. A *set system* over N is a pair $(N, \mathcal{F})$, where $\mathcal{F} \subseteq 2^N$ is a collection of subsets of N.

Definition 2.1 *A matroid is a set system* $(N, \mathcal{M})$ *with the following properties:*

(M1) $\emptyset \in \mathcal{M}$.

(M2) If $S \in \mathcal{M}$ *and* $T \subseteq S$, *then* $T \in \mathcal{M}$.

(M3) If $T, S \in \mathcal{M}$ *and* $|S| = |T| + 1$, *then there exists an* $i \in S \setminus T$ *such that* $T \cup \{i\} \in \mathcal{M}$.

We refer the reader to Welsh [9] for a detailed treatment of matroids. Throughout this paper we will often denote the matroid $(N, \mathcal{M})$ by $\mathcal{M}$. From now on we assume that the matroid $\mathcal{M}$ is *normal,* i.e., for every $i \in N$ there exists an $S \in \mathcal{M}$ such that $i \in S$. We present some examples of matroids.

Example 2.1 *For* $k \in \mathbb{N}$, $0 < k \leq n$, *the family* $\{S \subseteq N : |S| \leq k\}$, *of all subsets of* N *of size at most* k *is the uniform matroid* U_n^k. *In particular,* 2^N *is called the free matroid.*

Example 2.2 *If* $i, j \in N$, $i \neq j$, *the family* $\mathcal{M}_n (i \| j) = \{S \subseteq N : \{i, j\} \nsubseteq S\}$, *is a matroid.*

Example 2.3 *Let* E *be the set of edges of a graph* G *and let* $\mathcal{M} = \mathcal{M}(G)$ *be a family that consists of those subsets of* E *that do not contain a circuit of* G. *The matroid* $(E, \mathcal{M})$ *is called a graphic matroid.*

Definition 2.2 *A game on a matroid* $\mathcal{M}$ *is a function* $v : \mathcal{M} \to \mathbb{R}$ *with* $v(\emptyset) = 0$.

We denote by $\Gamma(\mathcal{M})$ the set of all games on the matroid $\mathcal{M} \neq \{\emptyset\}$. The set $\Gamma(\mathcal{M})$ is a vector space over $\mathbb{R}$. We also need to introduce the following definitions and results.

Definition 2.3 *For a given set system* $(N, \mathcal{M})$ *and* $S \subseteq N$*, a subset* $B \subseteq S$*,* $B \in \mathcal{M}$ *is called a basis of* S *if* $B \cup \{i\} \notin \mathcal{M}$ *for all* $i \in S \setminus B$*.*

Let $(N, \mathcal{M})$ be a matroid. Then B is a basis of S when B is a maximal feasible subset of S. A basis of the ground set N is called *basic coalition.* We denote by $\mathcal{B}$ the set of basic coalitions of $(N, \mathcal{M})$ and its cardinality by $b = |\mathcal{B}|$. The *rank* of the matroid $(N, \mathcal{M})$ is the cardinality of the basic coalitions. Then we have

$$\mathcal{B} = \{B \in \mathcal{M} : |B| = r\},$$

where r is the rank of $\mathcal{M}$. A maximal chain of $\mathcal{M}$ is a family of feasible coalitions

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{r-1} \subset S_r = B,$$

where $B \in \mathcal{B}$ and the cardinal $|S_k| = k$, for all $k = 0, 1, \ldots, r$. Note that the length of all the maximal chains of a matroid $\mathcal{M}$ is the rank of $\mathcal{M}$.

We can consider a matroid $\mathcal{M} \neq 2^N$ split into influence zones, where the basic coalitions are the summits of the different influence zones. Note that these influence zones are not disjoint.

Definition 2.4 *The influence set of* $S \in \mathcal{M}$ *is the set* $\mathcal{B}_S = \{B \in \mathcal{B} : S \subseteq B\}$, *of the basic coalitions that contain* S *and its influence worth is the quotient* b_S/b*, where* $b_S = |\mathcal{B}_S|$*.*

We denote $b_{\{i\}}$ by b_i when $S = \{i\}$. Let $(N, \mathcal{M})$ be a normal matroid. Then $\mathcal{M} = 2^N$ if and only if $b = 1$.

3 The τ-value for games on matroids

Let $\mathcal{M}$ be a matroid on $N = \{1, \ldots, n\}$ such that $\{i\} \in \mathcal{M}$ for all $i \in N$, and let $v \in \Gamma(\mathcal{M})$. The *upper vector* $M^v \in \mathbb{R}^n$ is given by

$$M_i^v = \frac{1}{b_i} \sum_{B \in \mathcal{B}_i} (v(B) - v(B \setminus i)).$$

If we consider the efficiency property on matroids then the payoff for the players must not exceed this value and therefore M^v is the utopia vector. Let $i \in N$ and we consider $S \in \mathcal{M}$ such that $i \in S$. If the other players in S obtain their utopia vectors, then the number $R^v(S, i) = v(S) - M^v(S \setminus i)$, is called the *remainder* for player i in the coalition S. The *lower vector* $m^v \in \mathbb{R}^n$ is defined by

$$m_i^v = \max_{\{S \in \mathcal{M} :\, i \in S\}} R^v(S, i).$$

This value is the minimal value expected by the player i and then is called also the minimal right vector.

Proposition 3.1 *If $v \in \Gamma(\mathcal{M})$, $d \in \Gamma(\mathcal{M})$ is an additive game and $\alpha \in \mathbb{R}$, then*

1. $M^{\alpha v+d} = \alpha M^v + d$,
2. $R^{\alpha v+d}(S,i) = \alpha R^v(S,i) + d_i$, *where $S \in \mathcal{M}$ and $i \in S$.*

Proof. 1. Let $i \in N$,

$$\begin{aligned} M_i^{\alpha v+d} &= \frac{1}{b_i} \sum_{B \in \mathcal{B}_i} ((\alpha v + d)(B) - (\alpha v + d)(B \setminus i)) \\ &= \frac{1}{b_i} \sum_{B \in \mathcal{B}_i} (\alpha v(B) - \alpha v(B \setminus i)) + d_i \\ &= \frac{\alpha}{b_i} \sum_{B \in \mathcal{B}_i} (v(B) - v(B \setminus i)) + d_i \\ &= \alpha M_i^v + d_i. \end{aligned}$$

2. Let $S \in \mathcal{M}$ and $i \in S$,

$$\begin{aligned} R^{\alpha v+d}(S,i) &= \alpha v(S) + d(S) - M^{\alpha v+d}(S \setminus i) \\ &= \alpha v(S) + d(S) - \alpha M^v(S \setminus i) - d(S \setminus i) \\ &= \alpha R^v(S,i) + d_i. \quad \square \end{aligned}$$

In the context of games on matroids, we define the *gap function* introduced by Driessen and Tijs [3].

Definition 3.1 *The gap function of $v \in \Gamma(\mathcal{M})$ is the new game $g^v : \mathcal{M} \to \mathbb{R}$ given by $g^v(S) = M^v(S) - v(S)$, for all $S \in \mathcal{M}$.*

Note that the maximal concession which a player i can give is the number

$$\lambda_i^v = \min_{\{S \in \mathcal{M} : i \in S\}} g^v(S),$$

and therefore we call λ^v the *concession vector*. It is easy to prove that the lower vector is the difference between the upper and the concession vector.

Proposition 3.2 *If $v \in \Gamma(\mathcal{M})$ then $m^v = M^v - \lambda^v$.*

Proposition 3.3 *Let $v \in \Gamma(\mathcal{M})$, d an additive game on $\mathcal{M}$ and $\alpha \in \mathbb{R}$. Then*

1. $g^{\alpha v+d}(S) = \alpha g^v(S)$ *for all $S \in \mathcal{M}$,*
2. $\lambda^{\alpha v+d} = \alpha \lambda^v$, *if $\alpha \geq 0$.*
3. $m^{\alpha v+d} = \alpha m^v + d$, *if $\alpha \geq 0$.*

Proof. 1. Let $S \in \mathcal{M}$, then

$$\begin{aligned} g^{\alpha v+d}(S) &= M^{\alpha v+d}(S) - \alpha v(S) - d(S) \\ &= \alpha M^{v}(S) + d(S) - \alpha v(S) - d(S) = \alpha g^{v}(S). \end{aligned}$$

2. Let $i \in N$, then

$$\begin{aligned} \lambda_i^{\alpha v+d} &= \min_{\{S\in\mathcal{M}:\, i\in S\}} g^{\alpha v+d}(S) = \min_{\{S\in\mathcal{M}:\, i\in S\}} \alpha g^{v}(S) \\ &= \alpha \min_{\{S\in\mathcal{M}:\, i\in S\}} g^{v}(S) = \alpha \lambda_i^{v}. \end{aligned}$$

3. This property follows from Propositions 3.1 and 3.2. □

Definition 3.2 *A game $v \in \Gamma(\mathcal{M})$ is quasi-balanced if it satisfies:*

1. $m^v \leq M^v$.
2. $\sum_{B\in\mathcal{B}} m^v(B) \leq \sum_{B\in\mathcal{B}} v(B) \leq \sum_{B\in\mathcal{B}} M^v(B)$.

The class of quasi-balanced games on matroids is denoted by $QB(\mathcal{M})$. We can also define this class using the gap function and the concession vector,

$$\begin{aligned} QB(\mathcal{M}) &= \left\{ v \in \Gamma(\mathcal{M}) : \lambda^v \geq 0,\ \sum_{B\in\mathcal{B}} m^v(B) \leq \sum_{B\in\mathcal{B}} v(B) \leq \sum_{B\in\mathcal{B}} M^v(B) \right\} \\ &= \left\{ v \in \Gamma(\mathcal{M}) : g^v(S) \geq 0,\ \forall S \in \mathcal{M},\ \sum_{B\in\mathcal{B}} g^v(B) \leq \sum_{B\in\mathcal{B}} \lambda^v(B) \right\}. \end{aligned}$$

Definition 3.3 *The core $C(v,\mathcal{M})$ of a game v on a matroid $\mathcal{M}$ is the set*

$$\{x \in \mathbb{R}^n : x(B) = v(B) \text{ for all } B \in \mathcal{B},\ x(S) \geq v(S) \text{ for all } S \in \mathcal{M}\}.$$

We prove now that the core elements of games on matroids are bounded by the upper and the lower vectors.

Theorem 3.4 *Let $v \in \Gamma(\mathcal{M})$ be a game such that $C(v,\mathcal{M}) \neq \emptyset$. Then for all $x \in C(v,\mathcal{M})$ we have $m^v \leq x \leq M^v$.*

Proof. If $x \in C(v,\mathcal{M})$ then $v(B) = x(B)$ and $v(B \setminus i) \leq x(B \setminus i)$ for all the basic coalitions. Hence

$$\begin{aligned} M_i^v &= \frac{1}{b_i} \sum_{B\in\mathcal{B}_i} (v(B) - v(B \setminus i)) \\ &\geq \frac{1}{b_i} \sum_{B\in\mathcal{B}_i} (x(B) - x(B \setminus i)) \\ &= \frac{1}{b_i} \sum_{B\in\mathcal{B}_i} x_i = x_i. \end{aligned}$$

Further,

$$\begin{aligned} m_i^v &= \max_{\{S \in \mathcal{M}: i \in S\}} R^v(S,i) \\ &= R^v(T,i) = v(T) - M^v(T \setminus i) \\ &\leq x(T) - x(T \setminus i) = x_i, \end{aligned}$$

where

$$T \in \arg \max_{\{S \in \mathcal{M}: i \in S\}} R^v(S,i). \quad \square$$

Corollary 3.5 *If* $v \in \Gamma(\mathcal{M})$ *is a game with* $C(v, \mathcal{M}) \neq \emptyset$ *then* $v \in QB(\mathcal{M})$.

The τ-value for a quasi-balanced game on a matroid is the unique efficient vector on the segment between m^v and M^v.

Definition 3.4 *Let* $v \in QB(\mathcal{M})$. *The* τ*-value of* v *is the vector defined by*

$$\tau^v = m^v + \alpha(M^v - m^v),$$

where $\alpha \in \mathbb{R}$ *is such that* $\sum_{i \in N} b_i \tau_i^v = \sum_{B \in \mathcal{B}} v(B)$.

This value is given by

$$\tau^v = \begin{cases} M^v, & \text{if } g^v(B) = 0 \text{ for all } B \in \mathcal{B}, \\ M^v - \dfrac{\sum_{B \in \mathcal{B}} g^v(B)}{\sum_{B \in \mathcal{B}} \lambda^v(B)} \lambda^v, & \text{if there is a } B \in \mathcal{B} \text{ with } g^v(B) \neq 0. \end{cases}$$

Tijs [7] proved that the τ-value is the unique value on $QB(2^N)$ which satisfies efficiency, restricted proportionality and the minimal right property. We study the axiomatization for games on matroids by using the restricted proportionality property: τ^v is proportional to M^v if $m_i^v = 0$ for all $i \in N$.

Theorem 3.6 *The* τ*-value on* $QB(\mathcal{M})$ *satisfies the individual rationality, the dummy player, the covariance and the restricted proportionality properties.*

Proof. Let $v \in QB(\mathcal{M})$, then we have $m_i^v \leq \tau_i^v \leq M_i^v$ for all $i \in N$.
Individual rationality: If $i \in N$ then

$$\tau_i^v \geq m_i^v = \max_{\{S:\, i \in S\}} R^v(S,i) \geq R^v(\{i\}, i) = v(i).$$

The dummy player property: If $i \in N$ is a dummy player then for all $B \in \mathcal{B}_i$ we obtain $v(B) - v(B \setminus i) = v(i)$ and therefore

$$\tau_i^v \leq M_i^v = \frac{1}{b_i} \sum_{B \in \mathcal{B}_i} (v(B) - v(B \setminus i)) = v(i).$$

Covariance: Let $v \in QB(\mathcal{M})$, $d \in \Gamma(\mathcal{M})$ an additive game and $\alpha > 0$. By Propositions 3.3 and 3.1 we obtain that the game $\alpha v + d \in QB(\mathcal{M})$ and the value $\tau^{\alpha v + d} = \alpha \tau^v + d$.
Restricted proportionality: It follows from $\tau^v = m^v + \alpha(M^v - m^v)$. $\square$

Theorem 3.7 *The τ-value is the unique value on $QB(\mathcal{M})$ which satisfies efficiency, covariance and the restricted proportionality property.*

Proof. The τ-value satisfies the above properties by Theorem 3.6. We only prove that is the unique one-point solution with these properties. Let Ψ be a value that satisfies the axioms and let $v \in QB(\mathcal{M})$. We consider the game $w = v - m^v$. From the covariance we obtain that $\Psi(w) = \Psi(v) - m^v$ and $\tau^w = \tau^v - m^v$. Thus we only need to prove that $\Psi(w) = \tau^w$. By Propositions 3.1 and 3.3, we have $M^w = M^v - m^v$ and $m^w = 0$. Then $\tau^w = \beta M^w$ and, by the restricted proportionality, $\Psi(w) = \alpha M^w$. Since Ψ and τ are efficient we obtain that $\alpha = \beta$. $\square$

The next results deal with relations between the τ-value and the core for games on matroids.

Proposition 3.8 *Let $v \in \Gamma(\mathcal{M})$ be a game such that $g^v(S) \geq 0$ for all $S \in \mathcal{M}$. If $g^v(B) = 0$ for all $B \in \mathcal{B}$, then $C(v, \mathcal{M}) = \{\tau^v\} = \{M^v\}$.*

Proof. For any $i \in N$ there exists a $B \in \mathcal{B}$ such that $i \in B$. Then

$$\lambda_i^v = \min_{\{S \in \mathcal{M}: i \in S\}} g^v(S) = 0,$$

for all $i \in N$ and the game $v \in QB(\mathcal{M})$. Since $m^v = M^v - \lambda^v$, we obtain $m^v = \tau^v = M^v$. Next, our hypothesis show that $M^v(B) = g^v(B) + v(B) = v(B)$ for all $B \in \mathcal{B}$ and also $M^v(S) = g^v(S) + v(S) \geq v(S)$ for all $S \in \mathcal{M}$. Therefore $M^v \in C(v, \mathcal{M})$. Finally, Theorem 3.4 implies that $C(v, \mathcal{M}) = \{M^v\}$. $\square$

Theorem 3.9 *Let $v \in QB(\mathcal{M})$ be a game such that $g^v(B') > 0$ for some $B' \in \mathcal{B}$. Then $\tau^v \in C(v, \mathcal{M})$ if and only if for all $S \in \mathcal{M}$,*

$$\left(\sum_{B \in \mathcal{B}} g^v(B)\right) \lambda^v(S) \leq \left(\sum_{B \in \mathcal{B}} \lambda^v(B)\right) g^v(S).$$

Proof. Suppose first that $\tau^v \in C(v, \mathcal{M})$. Then $\tau^v(B) = v(B)$ for all $B \in \mathcal{B}$, and $\tau^v(S) \geq v(S)$ for all $S \in \mathcal{M}$. The definition of the τ-value gives

$$\tau^v(S) \geq v(S) \Longleftrightarrow M^v(S) - \frac{\sum_{B \in \mathcal{B}} g^v(B)}{\sum_{B \in \mathcal{B}} \lambda^v(B)} \lambda^v(S) \geq v(S),$$

which is equivalent to

$$\left(\sum_{B \in \mathcal{B}} g^v(B)\right) \lambda^v(S) \leq \left(\sum_{B \in \mathcal{B}} \lambda^v(B)\right) g^v(S),$$

for all $S \in \mathcal{M}$. Conversely, the above inequalities imply that $\tau^v(S) \geq v(S)$ for all $S \in \mathcal{M}$. Further, the τ-value satisfies

$$\sum_{B \in \mathcal{B}} \tau^v(B) = \sum_{i \in N} b_i \tau_i^v = \sum_{B \in \mathcal{B}} v(B) \Longleftrightarrow \sum_{B \in \mathcal{B}} (\tau^v(B) - v(B)) = 0$$

and $\tau^v(B) - v(B) \geq 0$ for all $B \in \mathcal{B}$. It follows that $\tau^v(B) = v(B)$ for all $B \in \mathcal{B}$. $\square$

4 Special subclasses of games on matroids

In relation to the core and quasi-balanced games, 1-convex games were introduced in [3] and semiconvex games in [4]. We define analogous subclasses of games on matroids.

Definition 4.1 *A game $v \in \Gamma(\mathcal{M})$ is* 1-convex *with gap $g \geq 0$ if $g^v(B) = g$ for all $B \in \mathcal{B}$ and $g^v(S) \geq g$ for each $S \in \mathcal{M}$.*

The family of 1-convex games on a matroid $(N, \mathcal{M})$ with gap g is denoted by $C_g^1(\mathcal{M})$.

Theorem 4.1 *If $v \in C_g^1(\mathcal{M})$ then $v \in QB(\mathcal{M})$ and for all $i \in N$*

$$\tau_i^v = M_i^v - \frac{g}{r},$$

where r is the rank of the matroid $\mathcal{M}$. Moreover, $\tau^v \in C(v, \mathcal{M})$.

Proof. If a game v is 1-convex with gap $g \geq 0$, then $g^v(S) \geq g \geq 0$, for all $S \in \mathcal{M}$, and hence

$$\lambda_i^v = \min_{\{S \in \mathcal{M}: i \in S\}} g^v(S) = g,$$

for all $i \in N$. Since $|\{B : B \in \mathcal{B}\}| = b$ and $|B| = r > 1$ for every basic coalition, we obtain

$$\sum_{B \in \mathcal{B}} g^v(B) = bg,$$
$$\sum_{B \in \mathcal{B}} \lambda^v(B) = \sum_{B \in \mathcal{B}} rg = brg,$$

Then $\sum_{B \in \mathcal{B}} g^v(B) \leq \sum_{B \in \mathcal{B}} \lambda^v(B)$ for all $B \in \mathcal{B}$, and so $v \in QB(\mathcal{M})$. Its τ-value satisfies for every $i \in N$,

$$\tau_i^v = M_i^v - \frac{\sum_{B \in \mathcal{B}} g^v(B)}{\sum_{B \in \mathcal{B}} \lambda^v(B)} \lambda_i^v = M_i^v - \frac{bg}{brg} g = M_i^v - \frac{g}{r}.$$

We observe now that $g^v(B) = g$ for all $B \in \mathcal{B}$ implies $M^v(B) - v(B) = g$, and therefore

$$\tau^v(B) = M^v(B) - \sum_{i \in B} \frac{g}{r} = v(B) + g - g = v(B).$$

The game v satisfies $g^v(S) \geq g$ for all $S \in \mathcal{M}$, and we obtain also that

$$\begin{aligned} \tau^v(S) &= M^v(S) - \sum_{i \in S} \frac{g}{r} \\ &\geq v(S) + g - \frac{|S| g}{r} \\ &= v(S) + \frac{(r - |S|) g}{r} \\ &\geq v(S), \end{aligned}$$

using $r \geq |S|$ for every $S \in \mathcal{M}$. The conclusion is that $\tau^v \in C(v, \mathcal{M})$. $\square$

Definition 4.2 *A game on a matroid* $v \in \Gamma(\mathcal{M})$ *is semiconvex if for each* $i \in N$ *and* $S \in \mathcal{M}$ *such that* $i \in S$, *we have* $g^v(S) \geq g^v(i) \geq 0$.

The family of semiconvex games on a matroid $(N, \mathcal{M})$ is denoted by $SC(\mathcal{M})$. The imputation set $I(v, \mathcal{M})$ of a game $v \in \Gamma(\mathcal{M})$ is the set

$$\{x \in \mathbb{R}^n : x(B) = v(B) \text{ for all } B \in \mathcal{B},\ \ x_i \geq v(i) \text{ for all } i \in N\}.$$

Note that if $I(v, \mathcal{M}) \neq \emptyset$ then $\sum_{i \in B} v(i) \leq v(B)$ for all $B \in \mathcal{B}$.

Theorem 4.2 *If* $v \in SC(\mathcal{M})$ *and* $I(v, \mathcal{M}) \neq \emptyset$, *then* $v \in QB(\mathcal{M})$. *Moreover, if* $g^v(k) > 0$ *for some* $k \in N$, *then*

$$\tau_i^v = M_i^v - \frac{\sum_{B \in \mathcal{B}} g^v(B)}{\sum_{k \in N} b_k g^v(k)} g^v(i).$$

Otherwise, $\tau^v = M^v$.

Proof. If the game v is semiconvex then $g^v(S) \geq g^v(i) \geq 0$, for all $S \in \mathcal{M}$ such that $i \in S$. Hence

$$\lambda_i^v = \min_{\{S \in \mathcal{M} : i \in S\}} g^v(S) = g^v(i) = M_i^v - v(i),$$

for all $i \in N$. Since $I(v, \mathcal{M}) \neq \emptyset$, it follows that

$$\begin{aligned}
\sum_{B \in \mathcal{B}} \lambda^v(B) &= \sum_{B \in \mathcal{B}} \sum_{i \in B} \left(M_i^v - v(i)\right) \\
&= \sum_{B \in \mathcal{B}} \left(M^v(B) - \sum_{i \in B} v(i)\right) \\
&\geq \sum_{B \in \mathcal{B}} \left(M^v(B) - v(B)\right) \\
&= \sum_{B \in \mathcal{B}} g^v(B).
\end{aligned}$$

From this and the inequality $g^v(S) \geq 0$ for all $S \in \mathcal{M}$, we deduce that $v \in QB(\mathcal{M})$. To prove the formula of the τ-value we compute

$$\begin{aligned}
\sum_{B \in \mathcal{B}} \lambda^v(B) &= \sum_{B \in \mathcal{B}} \sum_{i \in B} g^v(i) \\
&= \sum_{i \in N} \sum_{B \in \mathcal{B}_i} g^v(i) \\
&= \sum_{i \in N} b_i g^v(i).
\end{aligned}$$

Suppose that $g^v(k) > 0$ for some $k \in N$. We can find $B \in \mathcal{B}$ with $k \in B$, and hence $g^v(B) \geq g^v(k) > 0$. Since $\lambda_i^v = g^v(i)$ for all $i \in N$, Definition 3.4 implies the formula. Otherwise, $\lambda_i^v = 0$ for all $i \in N$ and so $\tau^v = M^v$. $\square$

Let $v \in SC(\mathcal{M})$ such that $v(i) = 0$ for all $i \in N$. Then $g^v(i) = M_i^v$ and hence

$$\sum_{i \in N} b_i g^v(i) = \sum_{i \in N} b_i M_i^v = \sum_{B \in \mathcal{B}} M^v(B).$$

If $I(v, \mathcal{M}) \neq \emptyset$ then the τ-value satisfies

$$\begin{aligned} \tau_i^v &= M_i^v - \frac{\sum_{B \in \mathcal{B}} g^v(B)}{\sum_{B \in \mathcal{B}} M^v(B)} M_i^v \\ &= \frac{\sum_{B \in \mathcal{B}} (M^v(B) - g^v(B))}{\sum_{B \in \mathcal{B}} M^v(B)} M_i^v \\ &= \frac{\sum_{B \in \mathcal{B}} v(B)}{\sum_{B \in \mathcal{B}} M^v(B)} M_i^v. \end{aligned}$$

Example 4.1 *We consider a simple game $v : 2^N \to \{0,1\}$ with two players $i, j \in N$ such that coalitions with both players are not feasible and $v(S) = 0$ if $S \cap \{i,j\} = \emptyset$. Then $(v, \mathcal{M}_n(i\|j))$ is a simple game on the matroid defined in Example 2.2. Suppose further that the two basic coalitions $B_i = N \setminus j$ and $B_j = N \setminus i$ are winning. For this matroid we have: $r = n-1$, $b = 2$, $b_i = b_j = 1$ and $b_k = 2$ for all $k \in N \setminus \{i,j\}$. The coalitions of the veto players in B_i and B_j are respectively*

$$\begin{aligned} V_i &= \{k \in N \setminus \{i,j\} : v(B_i \setminus k) = 0\}, \\ V_j &= \{k \in N \setminus \{i,j\} : v(B_j \setminus k) = 0\}. \end{aligned}$$

First, we obtain the upper vector:

$$M_k^v = \begin{cases} 1, & \text{if } k \in (V_i \cap V_j) \bigcup \{i,j\}, \\ 1/2, & \text{if } k \in (V_i \cup V_j) \setminus (V_i \cap V_j), \\ 0, & \text{otherwise.} \end{cases}$$

Then the gap function is given by

$$g^v(S) = \begin{cases} (1/2)(|V_i \cap S| + |V_j \cap S|), & \text{if } v(S) = 1 \text{ or } S \cap \{i,j\} = \emptyset, \\ (1/2)(|V_i \cap S| + |V_j \cap S|) + 1, & \text{otherwise,} \end{cases}$$

for all $S \in \mathcal{M}_n(i\|j)$. Since $g^v(S) \geq g^v(k) \geq 0$ for each $k \in S$, we conclude that v is a semiconvex game. Moreover, $e_i + e_j \in I(\mathcal{M})$ and Theorem 4.2 *implies that* $v \in QB(\mathcal{M})$. *The* τ*-value is given by*

$$\tau_k^v = \begin{cases} \dfrac{2}{|V_i| + |V_j| + 2}, & \textit{if } k \in (V_i \cap V_j) \bigcup \{i, j\}, \\ \\ \dfrac{1}{|V_i| + |V_j| + 2}, & \textit{if } k \in (V_i \cup V_j) \setminus (V_i \cap V_j), \\ \\ 0, & \textit{otherwise.} \end{cases}$$

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